Solution by Zach Pence

Problem 1.23

**Problem Statement:** Prove that \((\sum_{k=1}^{n} a_kb_k)^2 = (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - \sum_{1 \leq k < j \leq n} (a_kb_j - a_jb_k)^2\) when \(a, b \in \mathbb{R}\).

**Proof.** Note that the equation we want to prove can be restated as

\[
(\sum_{k=1}^{n} a_kb_k)^2 = (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (a_kb_j - a_jb_k)^2.
\]

(1)

Our goal is to show that we can express the terms of (1) in such a manner that our expressions on both sides of the equals sign are identical. First, we will focus on simplifying the left-hand side of the equation which we can expand as

\[(a_1b_1 + a_2b_2 + ... + a_nb_n)(a_1b_1 + a_2b_2 + ... + a_nb_n).
\]

Through some algebra, we can multiply this out to

\[a_1^2b_1^2 + a_1b_1a_2b_2 + ... + a_1b_1a_nb_n + a_1b_1a_2b_2 + a_2^2b_2^2 + ... + a_2b_2a_nb_n + a_1b_1a_nb_n + a_2b_2a_nb_n + ... + a_n^2b_n^2,
\]

so we have

\[
(\sum_{k=1}^{n} a_kb_k)^2 = \sum_{k=1}^{n} a_k^2b_k^2 + \sum_{k=1}^{n} \sum_{j=k+1}^{n} a_kb_ja_jb_j.
\]

(2)

Now, moving on to the right-hand side of the equation, we can expand \((\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)\) to

\[(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2),
\]

which multiplies out to

\[a_1^2b_1^2 + a_1^2b_2^2 + ... + a_1^2b_n^2 + a_2^2b_1^2 + a_2^2b_2^2 + ... + a_2^2b_n^2 + ... + a_n^2b_1^2 + a_n^2b_2^2 + ... + a_n^2b_n^2.
\]

Thus, we have

\[
(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) = \sum_{k=1}^{n} a_k^2b_k^2 + \sum_{k=1}^{n} \sum_{j=k+1}^{n} a_k^2b_j^2 + \sum_{k=1}^{n} \sum_{j=k+1}^{n} a_j^2b_k^2.
\]

(3)

Now, we will move on to the last part of the right-hand side, where we see that

\[
\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (a_kb_j - a_jb_k)^2
\]

can be expanded as

\[(a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + ...a_{n-1}b_n - a_nb_{n-1})(a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + ...a_{n-1}b_n - a_nb_{n-1}).
\]

The result in summation notation is

\[
\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (a_kb_j - a_jb_k)^2 = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} a_k^2b_j^2 + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} a_j^2b_k^2 - 2 \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} a_kb_ja_jb_j.
\]

(4)

Recall that our goal is to obtain two identical expressions. We take (3)-(4) to get

\[
(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (a_kb_j - a_jb_k)^2 = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} a_k^2b_j^2 + 2 \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} a_kb_ja_jb_j,
\]

(5)

and since (2)=(5) the result is proved. \(\square\)