Solution by Hayden Harper
Problem 1.10

**Problem Statement:** If $a, b, c, d$ are rational and if $x$ is irrational, prove that $(ax + b)/(cx + d)$ is usually irrational. When do exceptions occur?

**Proof.** Suppose, for a contradiction, that $(ax + b)/(cx + d)$ is rational. Then,

$$\frac{ax + b}{cx + d} = \frac{p}{q},$$

where $p, q$ are integers without a common factor and $q \neq 0$. By rearrangement, we have that

$$(ax + b)q = (cx + d)p.$$

So,

$$axq - cxp = dp - bq.$$

By factoring, it is clear that

$$x = \frac{dp - bq}{aq - cp},$$

if and only if $aq - cp \neq 0$. In this case, Since $a, b, c, d, p, q$ are all rational, this implies that $x$ is rational, which contradicts our assumption on $x$. Therefore, it must be that $(ax + b)/(cx + d)$ is irrational.

If however, $aq - cp = 0$, then we have that

$$(aq - cp)x + (bq - dp) = 0.$$

Since $aq - cp = 0$, this implies that $bq - dp = 0$. Then we have that $aq = cp$ and $bq = dp$. Consider $aq = cp$. Multiplying by $d$ on both sides, we have that

$$aqd = cpd.$$

With substitution,

$$aqd = cbq,$$

which implies that

$$ad = bc.$$

In this case, $(ax + b)/(cx + d)$ is rational since no contradiction can be found. Therefore, $(ax + b)/(cx + d)$ is irrational unless $ad = bc$ in which case $(ax + b)/(cx + d)$ is rational, which is the only exception. \qed