Solution by Jared Polonitza
Problem 1.25

Prove Minowski’s inequality:

\[
\left( \sum_{k=1}^{n} \left( a_k + b_k \right)^2 \right)^{1/2} \leq \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} + \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2}
\]  \hspace{1cm} (1)

The proof will involve the Cauchy-Schwarz inequality (Theorem 1.23 in the text)

**Theorem 1.** Theorem 1.23 from the book, the Cauchy-Schwarz inequality states if \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) are arbitrary real numbers, we have,

\[
\left( \sum_{k=1}^{n} a_k b_k \right) \leq \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2}.
\]  \hspace{1cm} (2)

We now prove the main statement:

**Proof.** Since Minowski’s inequality is of the form \( x < y \), both sides are positive, therefore squaring both sides would not change the equation as \( x^2 < y^2 \) is still true. Take Minowski’s inequality and square both sides. The resulting equation is still equivalent to (1),

\[
\sum_{k=1}^{n} \left( a_k + b_k \right)^2 \leq \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} + \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2} \left( \sum_{k=1}^{n} a_k^2 \right) + 2 \left( \sum_{k=1}^{n} a_k b_k \right) + \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2}.
\]  \hspace{1cm} (3)

Expanding the left hand side of (3) yields,

\[
\sum_{k=1}^{n} a_k^2 + 2 \left( \sum_{k=1}^{n} a_k b_k \right) + \sum_{k=1}^{n} b_k^2.
\]  \hspace{1cm} (4)

By (1) equation (4) satisfies,

\[
\sum_{k=1}^{n} a_k^2 + 2 \left( \sum_{k=1}^{n} a_k b_k \right) + \sum_{k=1}^{n} b_k^2 \leq \sum_{k=1}^{n} a_k^2 + 2 \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) + \sum_{k=1}^{n} b_k^2.
\]  \hspace{1cm} (5)

This shows (3), therefore Minowski’s inequality must be true. \(\square\)