Quiz 8

Each problem is worth 2 points. Please show your work. No calculators or technology allowed, but it is OK
to leave work in relatively unsimplified form.

1. Calculate the integral of \( f(x, y) = x^2y \) over the ring illustrated below (i.e. integrate over the region
outside the inner circle but inside the outer circle).

   Since \( f(x, y) = -f(x, y) \), \( f(x, y) \) is odd
   in \( y \), and since the region
   is symmetric around the \( x \)-axis,
   \[
   \iint f(x, y) \, dA = 0
   \]

2. Integrate the function \( f(x, y) = 2x \) over the region below:

   \[
   \iint 2x \, dA = \int_A 2x \, dA + \int_B 2x \, dA
   \]
   \[
   = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y = \frac{1}{2}x}^{3x + 5} 2x \, dy \, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{y = \frac{1}{2}x}^{2 - 2x^{1/5}} 2x \, dy \, dx
   \]
   \[
   = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x \left( 3x + 5 - \frac{1}{2}x \right) \, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x \left( 2 - \frac{3x + 5}{2} \right) \, dx
   \]
   \[
   = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( 8x^2 + 10x^2 - x^2 \right) \, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( -4x^2 + 10x - x^2 \right) \, dx
   \]
   \[
   = \left[ \frac{8x^3}{3} + \frac{10x^2}{2} - \frac{x^3}{3} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \left[ \frac{8}{3} + \frac{40}{4} - \frac{1}{3} \right] - \left[ \frac{8}{3} + \frac{40}{4} - \frac{1}{3} \right] = 0
   \]
3. Calculate \( \iiint_W y \, dV \), where \( W \) is the region above \( z = x^2 + y^2 \) and below \( z = 5 \), and bounded by \( y = 0 \) and \( y = 1 \).

\[
\begin{align*}
\int_{y=0}^{1} \int_{x=0}^{\sqrt{5-y^2}} y \, dxdy &= \int_{y=0}^{1} \int_{x=0}^{\sqrt{5-y^2}} \frac{\sqrt{25-y^2}}{1+\frac{y}{2}} \, dx \, dy \\
&= \int_{0}^{1} \left( 5y - \frac{5}{3} y^3 - \frac{3}{5} y^5 \right) \, dy \\
&= 2 \int_{0}^{1} \left( 5y \sqrt{25-y^2} - \frac{5}{3} y^3 \sqrt{25-y^2} - \frac{5}{5} y^5 \sqrt{25-y^2} \right) \, dy \\
&= 2 \int_{0}^{1} \left( \frac{5}{2} \sqrt{25-y^2} \, dy + \frac{5}{6} \sqrt{25-y^2} \, dy - \frac{5}{2} \sqrt{25-y^2} \, dy \right)
\end{align*}
\]

4. Use the definition of volume to prove that the volume of a sphere of radius \( r \) is \( \frac{4}{3} \pi r^3 \).