Exam 2

For partial credit, please show your work. No calculators or technology allowed, but it is OK to leave work in relatively unsimplified form.

Problem 1 (15 pts) Contours. Consider the function \( f(x, y) = y - \sin x \).

a) Fill out the following on the axes to the right:

- Sketch the level lines \( f(x, y) = d \) for \( d = -1, 0, 1 \).
- Place a dot at the point \( P = (0, 0) \), and sketch the gradient vector \( \nabla f(P) \) at this point. (Focus on its direction, don’t worry about length.)
- Sketch the tangent line to the level curve \( f(x, y) = 0 \) at \( Q = (\pi/2, 1) \).

b) What is the average rate of change in the elevation of an ant who moves from \( P = (0, 0) \) to \( Q = (\pi/2, 1) \) along the surface of \( f \)?

c) Suppose the ant were stationed at the point on the surface corresponding to \( (x, y) = (\pi/2, 1) \) and wanted to go downhill as quickly as possible. In which direction should she head? Add a vector pointing in this direction to your contour plot.
Problem 2 (10 pts) Limits and continuity. Evaluate the following limits, or show that they do not exist.

a) \[ \lim_{(x,y) \to (0,0)} \cos(x+y)e^{-\frac{1}{x^2+y^2}}. \]

b) \[ \lim_{(x,y) \to (0,0)} \frac{x}{2x+y}. \]

Problem 3 (10 pts) Gradients and directional derivatives. Let \( f(x,y) = e^{xy} + x \cos y + 2y. \)

a) Calculate the gradient \( \nabla f. \)

b) Calculate the directional derivative \( D_v f(P), \) where \( v = (1,1) \) and \( P = (0,0). \)
Problem 4 (10 pts) Chain Rule.

a) Let \( f(x, y) = xy^2 + 2x \), where \( x = \sin(r + s) \) and \( y = sr^2 \). Calculate \( \partial f / \partial r \).

b) Suppose \( x^2 + y^2 - 2z^2 + 12x - 8z - 4 = 0 \). Calculate \( \partial z / \partial x \).

Problem 5 (10 pts) Tangent lines and planes. Let \( f(x, y) = y \ln x + x + y \).

a) Calculate the equation of the tangent line to the level curve \( f(x, y) = 1 \) at the point \( (1, 0) \).

b) Calculate the equation of the tangent plane to the graph of \( z = f(x, y) \) at the point \( (1, 0, 1) \).
Problem 6 (10 pts) Elementary integrals in two and three dimensions

a) Evaluate \( \int_{-1}^{1} \int_{-2}^{2} (x + y) \, dy \, dx \)

b) Evaluate \( \int_{0}^{1} \int_{2}^{3} \int_{-\pi/2}^{\pi/2} x \cos z \, dz \, dy \, dx \)

Problem 7 (10 pts) Changing the order of integration. Suppose you want to integrate some function \( f(x, y) \) over the following region:

Set up the integral in two ways: A) where you integrate first with respect to \( x \), then with respect to \( y \), and B) where you integrate first with respect to \( y \), then with respect to \( x \). (Note that you do not need to solve these integrals, just set them up.)

Method A (\( dxdy \))

Method B (\( dydx \))
Problem 8 (5 pts) Almost done! Reward yourself by drawing an elephant in lederhosen for five easy points. (Feel free to switch out lederhosen for the strange accoutrement of your choice.)

Problem 9 (10 pts) Identifying regions of integration. Sketch the region over which the following area or volume integrals are being taken:

A \[ \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx \]

B \[ \int_{-1}^1 \int_{-x^2}^{x^2} f(x, y) \, dy \, dx \]

Problem 10 (10 pts) Area and volume integrals over non-square regions

a) Find the area of the region R bounded by \( x = 0, x = \pi, \) and \( y = \sin x. \)

b) Find the volume of the tetrahedron whose vertices are \( (0, 0, 0), (2, 0, 0), (0, 4, 0), (0, 0, 8). \) (Hint: first sketch the region. Then use your sketch to help set up an appropriate integral.)