Chapter 14 Review

1. Let \( f(x, y) = x^2 + 2y^2 \), and \( P = (1/\sqrt{2}, 1/2) \).

   (a) Produce two plots, side by side. For the first plot, sketch the level lines \( f(x, y) = d \) for \( d = 1, 2, 3 \). (This is a 2-D plot.) For the second, sketch the graph of \( f(x, y) \). (This is a 3-D plot.) Add the point \( P \) to the first graph, and the point \( (P, f(P)) \) to the second.

   (b) Write down the equation for the tangent line to the level curve \( f(x, y) = 1 \), and add a sketch of this line to your first plot above. Then write down the equation for the tangent plane to the graph of \( f(x, y) \) at the point \( (P, f(P)) \), and add a sketch of this plane to your second plot above.

   (c) Calculate \( \nabla f(P) \) and adjoin this vector to the point \( P \) in your first plot above. Note \( \nabla f \) is perpendicular to the level line through \( P \). Can you find a vector perpendicular to the graph at the point \( (P, f(P)) \) in your second plot? (Hint: think of this graph as the level surface of some function, and take the gradient. What is the relevant function?)
2. Consider the graph $f(x, y)$ which produced the following contour plot:

(a) Sketch the gradient vectors at points $A$ and $B$. Don’t worry about length, but make sure you get the direction correct.

(b) Suppose $\mathbf{v} = (0, 1)$. Estimate $D_{\mathbf{v}} f(A)$ and $D_{\mathbf{v}} f(B)$.

(c) Suppose $\mathbf{v} = \frac{B - A}{\|B - A\|}$, and I asked you to estimate the directional derivative $D_{\mathbf{v}} f(A)$. Would the estimate

$$D_{\mathbf{v}} f(A) \approx \frac{f(B) - f(A)}{\|B - A\|}$$

be a reasonable way to do this? Why or why not?
3. Some Chain Rule exercises:

(a) Find $\partial f/\partial r$ and $\partial f/\partial s$ if $f(x, y, z) = xy + z^2$, $x = s^2$, $y = 2rs$, and $z = r^2$.

(b) Find $\frac{d}{dt} f(c(t))$ if $f(x, y) = y \ln x + 2^{xy}$ and $c(t) = (x(t), y(t)) = (e^t, 3)$.

(c) Find $\frac{\partial z}{\partial y}$ if $e^{xy} + \sin(xy) + y = 0$. 
4. Evaluate the following limits:

(a) \( \lim_{{(x,y) \to (0,0)}} \frac{xy}{\sqrt{x^2 + y^2}} \)

(b) \( \lim_{{(x,y) \to (0,0)}} \tan(x^2 + y^2) \tan^{-1} \left( \frac{1}{x^2 + y^2} \right) \)

(c) \( \lim_{{(x,y) \to (2,1)}} e^{x^2 - y^2} \)