Chapter 17: Fundamental Theorems of Vector Calculus

Key points:

1. **Green’s Theorem:** Let $D \subseteq \mathbb{R}^2$ be a region whose boundary is a simple, closed curve, oriented counterclockwise. Let $F(x, y) = (F_1(x, y), F_2(x, y))$ be a differentiable vector field. Then

   $$\oint_{\partial D} F \cdot ds = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

   The left hand side is called the **circulation** of $F$ around $\partial D$. Note that the right hand side would be zero if $F$ were conservative. Thus the circulation measures to what extent the field fails to be conservative.

2. The **curl** of a vector field $F = (F_1, F_2, F_3)$ is defined as

   $$\text{curl}(F) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left( -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} \right), \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

   Symbolically, we write $\text{curl}(F) = \nabla \times F$, where $\nabla$ is the gradient operator,

   $$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

   Note that the $z$–component of the curl is exactly what shows up in the right hand side of Green’s Theorem.

3. **Stoke’s Theorem:** Let $S \subseteq \mathbb{R}^3$ be an oriented surface in $\mathbb{R}^3$ with smooth, simple boundary curves. Then for any differentiable vector field $F$,

   $$\oint_{\partial S} F \cdot ds = \iint_S \text{curl}(F) \cdot dS.$$

   Note that conservative vector fields have zero curl. Thus Stoke’s Theorem confirms that the closed path integral of a conservative vector field is zero.

Problems:

1. Use Green’s Theorem to show that the area enclosed by a closed curve $C$ is given by

   $$\text{Area enclosed by } C = \frac{1}{2} \oint_C xy - ydx.$$ 

2. Use Green’s Theorem to evaluate the line integral

   $$\oint_C F \cdot ds,$$

   where $F = (xy^2, x)$ and $C$ is the unit circle, oriented counterclockwise.

3. One way to think about the value of the curl is that it measure how much a small paddle placed at a point would rotate. The following figures all have constant curl. Look at the picture and try to visualize what a small paddle placed at each point would do.

4. Verify Stoke’s Theorem for $F = (-y, 2x, x + z)$ and $S$ the upper hemisphere of the unit sphere, with outward pointing normals.