Section 16.4: Surface Integrals of Scalar Valued Functions

Key points:

1. A parameterized surface in \( n \) dimensions is just the image of a smooth function \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^n \). A parameterized surface in three dimensions thus has the form

\[
S = \{ G(u, v) : (u, v) \in D \subset \mathbb{R}^2 \},
\]

where \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a smooth (continuously differentiable) function. Alternatively, we can write

\[
S = \{ \langle x(u, v), y(u, v), z(u, v) \rangle : (u, v) \in D \subset \mathbb{R}^2 \},
\]

where \( x, y, \) and \( z \) are smooth functions from \( \mathbb{R}^2 \rightarrow \mathbb{R} \).

2. The tangent and normal vectors of \( S \) at a point \( P = G(u, v) \) are given by

\[
T_u = \frac{\partial G}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad T_v = \frac{\partial G}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle,
\]

and

\[
\mathbf{n} = \mathbf{n}(u, v) = T_u \times T_v.
\]

In general, \( T_u \) and \( T_v \) need not be orthogonal. If \( \mathbf{n}(u, v) \neq 0 \), the parameterization is called regular at \( (u, v) \).

3. The quantity \( ||\mathbf{n}(u, v)|| \) is called a distortion factor. It captures the extent to which \( G \) magnifies or contracts area. Specifically, for a small path of area \( D_i \) around a point \( (u_i, v_i) \), let \( S_i = G(D_i) \). Then

\[
\text{Area}(S_i) \approx ||\mathbf{n}(u_i, v_i)|| \text{Area}(D_i).
\]

It follows that the area of a surface \( S \) is given by

\[
\text{Area}(S) = \iint_D ||\mathbf{n}(u, v)|| \, du \, dv.
\]

4. The above formula allows us to define a surface integral over \( S \):

\[
\iint_S f(x, y, z) dS = \iint_D f(G(u, v)) ||\mathbf{n}(u, v)|| \, du \, dv.
\]

5. The book provides some useful formulas for cylinders, spheres, and graphs:

- Cylinder of radius \( R \):

\[
G(\theta, z) = \langle R \cos \theta, R \sin \theta, z \rangle
\]

\[
\mathbf{n} = R \langle \cos \theta, \sin \theta, 0 \rangle
\]

\[
dS = R \, d\theta \, dz
\]

- Sphere of radius \( R \):

\[
G(\theta, \phi) = \langle R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \theta \rangle
\]

\[
\mathbf{n} = R^2 \sin \phi \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle
\]

\[
dS = R^2 \sin \phi \, d\theta \, d\phi
\]
Graph of $z = g(x, y)$:

\[
G(x, y) = (x, y, g(x, y)) \\
\mathbf{n} = (-g_x, -g_y, 1) \\
dS = \sqrt{1 + g_x^2 + g_y^2} \, dx \, dy
\]

Problems:

(\# 3 and 4 in text):

1. Let $G(u, v) = (2u + 1, u - v, 3u + v)$. Show that $G$ parameterizes the plane $2x - y - z = 2$.

2. Calculate $T_u$, $T_v$, and $\mathbf{n}(u, v)$.

3. Find the area of the $S = G(D)$, where $D = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 1.\}$

4. Now find the area of $S = G(D)$ when $D = \{(u, v) : u^2 + v^2 \leq 1, u \geq 0, v \geq 0.\}$

5. For the latter $D$, calculate $\iint_S (x - y) dS$. (Hint: use polar coordinates.)