Section 16.3: Conservative Vector Fields

Key points:

1. In the last section we were introduced to the idea of a conservative vector field, namely, a vector field $F(x, y, z): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which is the gradient of some function $V: \mathbb{R}^3 \rightarrow \mathbb{R}$. The function $V$ is called a potential for the vector field $F$. The terminology originates in physics, where phrases like “potential energy” might be familiar to you, and ultimately trace back to the same core idea of a potential. Gravity and electromagnetism are two forces that can often be described in terms of a potential.

2. The wonderful news about conservative vector fields is that they allow us to solve path integrals via a theorem analogous to the Fundamental Theorem of Calculus: for a vector field $F$ with potential $V$, the path integral along any path connecting points $P$ and $Q$ is

$$\int_c F \cdot ds = V(Q) - V(P).$$

Note that there are two immediate implications of this fact:

- a path integral of a conservative vector field from $P$ to $Q$ doesn’t actually depend on the particular path chosen, i.e. is path independent.
- the path integral of a conservative vector field along any closed path (one that starts and stops at the same place) is zero.

3. How do you check when a vector field is conservative? There are a couple of ways:

- On an open, connected domain, path independence guarantees that a field is conservative.
- On a simply connected domain, it is sufficient to check the cross partial relations: if $F = (F_1, F_2, F_3)$, check that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$ 

Problems:

1. (#1) Let $V(x, y, z) = xy \sin(yz)$ and $F = \nabla V$. Evaluate $\int_c F \cdot ds$, where $c$ is any path from $(0, 0, 0)$ to $(1, 1, \pi)$.

2. (# 9) Find a potential function for $F = (y^2, 2xy + e^z, ye^z)$.

3. (# 11) Is it possible to find a potential field for $F = (\cos(xz), \sin(yz), xy \sin z)$? How do you know?

4. (# 18) Evaluate $\int_c \sin x dx + z \cos y dy + \sin y dz$ were $c$ is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.