Section 16.2: Line Integrals

Key points:

1. The good news of this section is that the only tools you’ll need are ones you’re already familiar with: dot products, vector valued functions and functions with vector inputs, path parameterizations, and anti-derivatives.

2. This section deals with two kinds of integrals: scalar line integrals and vector line integrals. The nomenclature is somewhat confusing, however, because both ultimately yield a scalar answer. Here are the formulas:

   scalar line integral: \[ \int_C f(x, y, z) \, ds = \int_a^b f(c(t)) \| c'(t) \| \, dt \]

   vector line integral: \[ \int_C \mathbf{F}(x, y, z) \cdot \, ds = \int_a^b \mathbf{F}(c(t)) \cdot c'(t) \, dt \]

   Note that in both cases, the curve \( C \) is parameterized by the vector valued function \( \mathbf{c} : \mathbb{R} \rightarrow \mathbb{R}^n \) (where \( n \) is usually three in this book.) If a curve starts at the same place it stops, it is called closed.

3. The reason we incorporate a dot-product into the vector line integral formula is motivated by applications, mostly in physics. For example, the vector integral formula can be used to quantify how much work you do climbing a mountain, where the work is only a function of the vertical component.

4. The curve \( C \) is called oriented if the direction in which the curve is traversed is significant. The orientation of curves is significant for the vector integral formula, but not for the scalar integral formula. An important (but easy) result is that the value of a vector valued integral changes sign if the underlying curve is traversed in the opposite direction.

Problems:

1. Sketch the vector field \( \mathbf{F}(x, y) = \langle 0, -1 \rangle \). Draw a few paths over this sketch. Estimate the sign of the vector integral of \( \mathbf{F} \) over the paths you sketched.

2. (# 10) Compute \( \int_C f(x, y) \, ds \) for \( f(x, y) = y^4/x^7 \), \( y = x^4/4 \) for \( 1 \leq x \leq 2 \).

3. (# 20) Compute \( \int_C \mathbf{F}(x, y) \cdot \, ds \) for \( \mathbf{F}(x, y) = \langle 4, y \rangle \) over the quarter circle \( x^2 + y^2 = 1 \) with \( x \leq 0, y \leq 0 \), oriented counter clockwise.