Section 14.7: Optimization

Key points:

1. In the real world, it is often important to find the point \((a, b)\) where a function \(f(x, y)\) achieves a maximum or a minimum. Examples:
   - Suppose \(f(x, y)\) represents profit for a business that produces \(x\) units of one thing and \(y\) units of another. The business owner would like to know what values of \(x\) and \(y\) maximize \(f\).
   - Suppose \(f(x, y)\) represents the amount of carbon dioxide produced by an engine with two design parameters, \(x\) and \(y\). An environmental engineer would like to minimize \(f\).

2. Just as in single variable calculus, if \(f(x, y)\) is a “nice” function (continuous, differentiable), then candidate points for maxima and minima are the critical points, defined as points \((a, b)\) where
   - \(f_x(a, b) = 0\) or \(f_x(a, b)\) does not exist, and
   - \(f_y(a, b) = 0\) or \(f_y(a, b)\) does not exist

3. To see whether a critical point is a max or a min, we look at the discriminant, defined as
   \[
   D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.
   \]
   If \((a, b)\) is a critical point, then
   - if \(D(a, b) > 0\) and \(f_{xx}(a, b) > 0\), then \(f(a, b)\) is a local minimum.
   - if \(D(a, b) > 0\) and \(f_{xx}(a, b) < 0\), then \(f(a, b)\) is a local maximum.
   - if \(D(a, b) < 0\) then \(f(a, b)\) is a saddle point.
   - if \(D(a, b) = 0\), the test is inconclusive.

4. Sometimes we need to find maxima and minima subject to some constraint. There are two key facts:
   - a continuous function \(f(x, y)\) will assume both a maximum and a minimum on a closed region \(D\)
   - this maximum and minimum occur either at critical points in the interior of \(D\), or on the boundary of \(D\).

Problems:

1. Find the critical points of \(f(x, y) = 8y^4 + x^2 + xy - 3y^2 - y^3\).
2. For each of the above, try to determine if the critical points are local maxima, local minima, saddle points.
3. Find the critical points of the function \(f(x, y) = xye^{-x^2-y^2}\).
4. Determine the global extreme values of \(f(x, y) = x + y\) on the region \(0 \leq x \leq 1, 0 \leq y \leq 1\).
5. Repeat with \(f(x, y) = e^{-x^2-y^2}\) on \(x^2 + y^2 \leq 1\).