Section 14.3: Partial Derivatives

Key points:

1. The *partial derivatives* of a function \( f(x, y) \) is defined using exactly the same limit idea that you saw in first semester calculus:

\[
\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}
\]

\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}
\]

2. On a practical level, one computes a partial with respect one variable by treating all the other variables as constants, and using the ordinary differentiation rules from first semester calculus.

3. Just as “the derivative” has an interpretation as “slope of tangent” in 1-D calculus, so the partial derivative has an interpretation as slope. The situation is complicated by the fact that functions of more than one variable have *many* slopes at a point. (Think of standing on the slope of a mountain and asking “what is the slope”? There are many answers, depending on which direction you start walking.) But there is a simple interpretation: the partial \( f_x(a, b) \) can be interpreted as the rate at which you go up or down if, starting at the point on the mountain corresponding to \((a, b)\), you start walking in the positive \(x\)– direction (mutatis mutandis for the partial \( f_y(a, b) \)).

4. Another way to articulate the same interpretation is to think about the slope of trace curves. Recall that a trace curve to a graph is the intersection of the graph with a plane. The partials give the slopes of trace curves, when the intersections plane is either perpendicular to the \(x\)– axis (\( f_y(a, b) \)) or perpendicular to the \(y\)– axis (\( f_x(a, b) \)).

5. Recall that in Calc I you learned the formula \( f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \). (This was called “linear approximation”. ) Since partial derivatives are defined using the same limit ideas as regular derivatives, this formula applies to functions of several variables, as well. In this setting, the formulas look like:

\[
f(x_0 + \Delta x, y_0) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x
\]

\[
f(x_0, y_0 + \Delta y) \approx f(x_0, y_0) + f_y(x_0, y_0)\Delta y
\]

6. Higher order partials are just partials of partials (just like higher order derivatives in 1-D calculus.)

Problems:

1. Find \( \frac{\partial}{\partial x}(x + y)e^{2x+3y} \)

2. Consider the line formed by the intersection of \( f(x, y) = x^2y^3 \) with the plane \( x = 2 \). Find the rate at which the height of this line is increasing at the point \( (2, 3) \).

3. Estimate the partials of the function for which this is the contour map at the points \( A \) and \( B \).