Section 13.2: Calculus of Vector Valued Functions

Key points:

• All of the calculus you learned in Math 180 and Math 181 can be applied directly to vector valued functions. The only thing you really need to keep in mind is that operations proceed componentwise. For example, if you have
  \[ r(t) = \langle x(t), y(t), z(t) \rangle, \]
  then
  \[ r'(t) = \langle x'(t), y'(t), z'(t) \rangle, \]
  while
  \[ \int_a^b r(t) = \left\langle \int_a^b x(t), \int_a^b y(t), \int_a^b z(t) \right\rangle. \]

• One thing that does become confusing in this section is keeping tracking of which functions are vector valued and which functions are scalar valued. For example, the text states that the chain rule is given by
  \[ \frac{d}{dt} r(g(t)) = g'(t)r'(g(t)). \]
  Is it clear that \( g(t) \) must be scalar valued? Why?

• The text provides various formulas for differentiating the various kinds of products we’ve seen. Specifically:

  Scalar time vector
  \[ \frac{d}{dt} f(t)r(t) = f(t)r'(t) + f'(t)r(t) \]

  Dot product
  \[ \frac{d}{dt}(r_1(t) \cdot r_2(t)) = r_1(t) \cdot r_2'(t) + r_1'(t) \cdot r_2(t) \]

  Cross product
  \[ \frac{d}{dt}(r_1(t) \times r_2(t)) = [r_1(t) \times r_2'(t)] + [r_1'(t) \times r_2(t)]. \]

The derivation of these formulas follows directly from writing out the relevant product and applying the one-dimensional product rule.

• If \( r(t) \) is a vector valued function, you can think of \( r \) as tracing out a one-dimensional surface in \( \mathbb{R}^3 \) as \( t \) changes values. With this image in mind, the derivative \( r'(t) \) can be interpreted as the velocity vector at time \( t \). Its magnitude and direction indicate the rate and orientation of the change of position at time \( t \).

• Of course the vector \( r(t) \) itself doesn’t “lie on the curve”. To actually write down the equation of the line tangent to the graph of a function \( r(t) \) at a point corresponding to \( t = t_0 \), you need to translate the velocity vector so it’s tail lies at \( r(t) \). This yields the formula for the tangent line:
  \[ L(t) = r(t_0) + t r'(t_0). \]
  Think of \( L(t) \) as the best linear approximation to \( r(t) \) at \( t = t_0 \).

• The fundamental theorem for vector valued functions states that if \( r(t) \) is continuous and \( R(t) \) is the anti-derivative of \( r(t) \), then
  \[ \int_a^b r(t) \, dt = R(b) - R(a). \]

Problems:

1. (HW # 6) Evaluate \( \lim_{t \to 0} \frac{r(t)}{t} \) for \( r(t) = \langle \sin t, 1 - \cos t, -2t \rangle \).

2. (HW # 10) Compute the derivative of \( b(t) = \langle e^{3t-4}, e^{6-t}, (t+1)^{-1} \rangle \).
3. (HW # 14) Sketch the curve $\mathbf{r}(t) = (1 - t^2, t)$ for $-1 \leq t \leq 1$. Compute the tangent vector at $t = 1$ and add it to the sketch.

4. (HW # 34) Find a parameterization of the tangent line at $s = 1$ for the curve $\mathbf{r}(s) = (\ln s)i + s^{-1}j + 9sk$.

5. (HW # 42) Evaluate

$$\int_0^1 \left( te^{-t^2}i + t \ln(t^2 + 1)j \right).$$