Section 12.5: Planes in $\mathbb{R}^3$

Key points:

- A plane is uniquely determined by a point $P_0 = (x_0, y_0, z_0)$ and a normal vector $n = (a, b, c)$.
- Given $P_0$ and $n$, the equation of a plane can be given in any of the three following forms:
  \[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (1) \]
  \[ ax + by + cz = d \quad (2) \]
  \[ n \cdot (x, y, z) = d. \quad (3) \]
- What is $d$? Some observations:
  - By comparing the first and second of the forms above, we see that $d = ax_0 + by_0 + cz_0 = n \cdot P$.
  - If the plane passes through the origin, then $d = 0$.
  - If you start with an equation of the form (2) or (3) and change $d$ while keeping everything else the same, you get a family of parallel planes.
- Three points determine a plane. If you are given three points $P$, $Q$, and $R$ and told to find the equation of the plane through them, first find a normal vector, and then use one of the above equations. To do this, use the formula
  \[ n = \overrightarrow{PQ} \times \overrightarrow{PR}. \]
- The trace of a plane is the intersection of the plane with some other plane (generally a coordinate plane.) To find the trace of a plane in the $xy$–plane, set $z = 0$ in the equation of the plane. (Mutatis mutandis for the other coordinate planes.)

Problems:

1. Sketch the plane $x = 1$. On the same axis, also the plane $x = 2$. Are these planes oriented? Are you surprised?
2. (HW #2) Find the equation of the plane whose normal vector is $(-1, 2, 1)$ and which passes through point $P = (3, 1, 9)$. Can you find another point that lies on this plane?
3. (HW # 20) Find the equation of the plane through the three points $P = (2, 0, 0)$, $Q = (0, 4, 0)$, and $R = (0, 0, 2)$.
4. Find the equation of the plane that contains the lines $r_1(t) = (2, 1, 0) + t(1, 2, 3)$ and $r_2(t) = (2, 1, 0) + t(3, 1, 8)$.
5. (HW #34) Find the intersection of the line $r(t) = (2, -1, -1) + t(1, 2, -4)$ with the plane $2x + y = 3$. 

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