Exam 2

Please show your work. Each subproblem is worth 5 points.

Problem 1  **Graphical understanding of continuity, differentiability, and asymptotes.** Let $f(x)$ denote the function whose graph is depicted in the image below.

![Graph of $f(x)$](image)

a) Identify all the points $x$ at which $f(x)$ has a jump discontinuity.

$x = 0$

b) Identify all the points $x$ at which $f(x)$ has a removable discontinuity.

$x = -3, 2$

c) Identify all the points $x$ at which $f(x)$ has an infinite discontinuity.

$x = 6$

d) Identify all the points $x$ at which $f(x)$ is not differentiable.

$x = -3, 0, 2, 5, 6$

e) Does $f(x)$ have any vertical or horizontal asymptotes? If so, what are they?

*Vertical:* $x = 6$

*No obvious horizontal asymptotes.*
Problem 2  EVALUATING LIMITS. Evaluate each of the following limits. Please show your work.

a)  \[ \lim_{{x \to \pi/4}} \frac{\tan x}{x} \]

\[ = \frac{\tan (\pi/4)}{\pi/4} \cdot \frac{\pi/4 - 1}{\pi} \]

\[ = \frac{1}{\pi} \cdot \frac{\pi/4 - 1}{\pi} \]

\[ = \frac{1}{4} - \frac{1}{2} \]

b)  \[ \lim_{{x \to 6}} \frac{x - 6}{x^2 - 36} \]

\[ = \lim_{{x \to 6}} \frac{(x - 6) / (x - 6)}{x - 6 / (x - 6)} \]

\[ = \lim_{{x \to 6}} \frac{1}{x - 6} \]

\[ = \frac{1}{12} \]

c)  \[ \lim_{{x \to 0}} \frac{2x}{\sin 3x} \]

\[ = \lim_{{x \to 0}} \frac{2}{3} \cdot \frac{3x}{\sin 3x} \]

\[ = \frac{2}{3} \cdot \frac{3}{3} \cdot \lim_{{x \to 0}} \frac{x}{\sin x} \]

\[ = \frac{2}{3} \]

d)  \[ \lim_{{x \to \infty}} \frac{2x^3 + 3x + 1}{2x + 1} \]

\[ = \lim_{{x \to \infty}} \frac{x^3 \left[ 2 + \frac{3}{x} + \frac{1}{x^3} \right]}{2 \cdot \frac{x}{x}} \]

\[ = \lim_{{x \to \infty}} \left[ 2 + \frac{3}{x} \right] \]

\[ = 2 \]

e)  \[ \lim_{{x \to \infty}} \frac{3x^6 + 1}{6x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1} \]

\[ = \lim_{{x \to \infty}} \frac{3 \cdot \frac{1}{x^6}}{6 \cdot \frac{1}{x^6} + 5 \cdot \frac{1}{x^5} + 4 \cdot \frac{1}{x^4} + 3 \cdot \frac{1}{x^3} + 2 \cdot \frac{1}{x^2} + \frac{1}{x} + 1 \cdot \frac{1}{x^6}} \]

\[ = \frac{3}{6} = \frac{1}{2} \]
Problem 3 Intermediate Value and Squeeze Theorems

a) Use the Intermediate Value Theorem to show that the function \( f(x) = 3^x - 2 \) has a root in the interval \((0, 1)\).

\[
\begin{align*}
\Delta(0) &= 3^0 - 2 = 1 - 2 = -1 \\
\Delta(1) &= 3^1 - 2 = 3 - 2 = 1
\end{align*}
\]

Since \( \Delta(x) \) is continuous and changes sign in \([0,1]\), by the Intermediate Value Theorem, 

b) Use the Squeeze Theorem to show that \( \lim_{x \to 0} x^2 \cos(\ln x) = 0 \).

\[
\begin{align*}
\Delta(x) &= x^2 \cos(\ln x), \\
\Phi(x) &= x^2, \\
\Psi(x) &= -x^2. \\
\end{align*}
\]

Then we have:

\[
\begin{align*}
\gamma(x) &= -\frac{x^2}{\cos(\ln x)} \\
\Phi(x) &= x^2, \\
\Psi(x) &= -x^2. \\
\end{align*}
\]

Problem 4 Definition of Derivative

a) Write down the mathematical definition of \( f'(a) \), the derivative of \( f \) at the point \( a \). (Note: we gave two definitions in class. Either one is fine.)

\[
\begin{align*}
\gamma'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\
&= \end{align*}
\]

b) Use your definition above to find \( f'(1) \) for \( f(x) = x^2 + 2 \).

\[
\begin{align*}
\gamma'(1) &= \lim_{x \to 1} \frac{(x^2 + 2) - (1^2 + 2)}{x - 1} \\
&= \lim_{x \to 1} \frac{x - 1}{x - 1} \\
&= \end{align*}
\]

1) 

2) 

3) 

c) Use your definition above to find the function \( f'(x) \) if \( f(x) = 3x - 7 \).

\[
\begin{align*}
\gamma'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(3(x + h) - 7) - (3x - 7)}{h} \\
&= \frac{3h}{h} = \end{align*}
\]
Problem 5  **MISCELLANEOUS.**

**a)** Suppose the graph of \(f(x)\) is as shown on the left. Sketch the graph of \(f'(x)\) on the right. (Note: you don’t need to be super accurate: I’ll most check for qualitative features. Focus on where \(f'(x)\) is positive, where it’s negative, and where it’s zero.)

![Graphs of f(x) and f'(x) captions]

**b)** Use the power rule to find \(\frac{d}{dx} x^{999}\)

\[
\frac{d}{dx} x^{999} = 999x^{998}
\]

**c)** Suppose the tangent line to the graph of \(f(x)\) at \(x = 2\) is \(7x + 3\). Find \(f(2)\) and \(f'(2)\).

\[\text{Slope} \ f' \ \text{at} \ x = 1 \ \text{is} \ m = 7. \ \text{Thus} \quad f'(2) = 7\]

\[f(2) = 7 \cdot 2 + 3 = 17\]

**d)** Find the equation of the tangent line to \(f(x) = 3x^2 + 1\) at \(x = 1\).

\[\text{Here} \quad f'(x) = 6x, \quad f'(1) = 6, \quad f(1) = 4. \quad \text{So:} \quad y - 4 = 6(x - 1)\]

**e)** True or false: if a function \(f(x)\) is differentiable at \(c\), then it must also be continuous at \(c\). Explain your answer.

(True). In order for \( \lim_{{h \to 0}} \frac{f(c+h) - f(c)}{h} \) to be well defined, the numerator must go to zero. This can only happen if \( \lim_{{h \to 0}} f(c+h) - f(c) \), which is the definition of continuity.