Exam 1

Please show your work. Each subproblem is worth 5 points.

Problem 1  TRIGONOMETRIC FUNCTIONS.

For the problems below, please refer to the following figure:

a) Sketch radial segments that correspond to the angles \( \theta_1 = \pi/4 \) and \( \theta_2 = -4\pi/3 \) radians. Label these angles.

b) From the point where your segments intersect the circle, draw verticals that connect to the \( x \)-axis. This process should produce right triangles. Label the vertical and horizontal legs of the first triangle \( x_1 \) and \( y_1 \), respectively. Do the same for the second triangle, using labels \( x_2 \) and \( y_2 \).

c) Below, indicate the sign of each of these \( x_i \)'s and \( y_i \)'s by circling the appropriate word:
   \( x_1 \)  positive/negative
   \( y_1 \)  positive/negative
   \( x_2 \)  positive/negative
   \( y_2 \)  positive/negative

d) Without doing any calculations, say whether \( \tan \theta_1 \) is positive or negative, and explain why. Do the same for \( \tan \theta_2 \). (Use the definition of \( \tan \) and your answer to the previous problem.)

\[
\tan \theta_1 = \text{positive} \quad (\text{since} \quad \frac{y_1}{x_1} = \frac{\text{pos}}{\text{pos}} = \text{positive}).
\]

\[
\tan \theta_2 = \text{negative} \quad (\text{since} \quad \frac{y_2}{x_2} = \frac{\text{pos}}{\text{neg}} = \text{neg-sive}).
\]

e) Finally, say what \( \tan \theta_1 \) and \( \tan \theta_2 \) actually are.

\[
\tan \theta_1 = 1
\]

\[
\tan \theta_2 = -\frac{\sqrt{3}}{2}
\]

\[
\text{Use 45-45-90 triangle}
\]

\[
\text{Use 30-60-90 triangle}
\]
Problem 2  LOGARITHMS AND EXPONENTS

The questions that follow will ask you plot some things on the axes below:

a) Sketch the graph of \( f(x) = 3^x \) on the axes above.

b) Sketch the graph of \( g(x) = \log_3(x) \) on the axes above.

c) What are the domain and range of \( f(x) \) and \( g(x) \)? Express your answers in interval notation.

\[
\begin{align*}
\text{Dom. } f & : (-\infty, \infty) \\
\text{Range } f & : (0, \infty) \\
\text{Dom. } g & : (0, \infty) \\
\text{Range } g & : (-\infty, \infty)
\end{align*}
\]

d) Suppose \( 3^x = 2 \). What is \( (3^{x+1})^2 \)?

\[
(3^{x+1})^2 = 3^{2(x+1)} = 3^{2x+2} = 3^2 \cdot 3^2 = 9 \cdot 9 = 81
\]

e) Find \( \log_3(27^{1/3}) \)

\[
\begin{align*}
27 &= 3^3 \\
\text{So, } 27^{1/3} &= (3^3)^{1/3} = 3 \\
\text{Thus, } \log_3 27^{1/3} &= \log_3 3 = 1
\end{align*}
\]
Problem 3  **Average and Instantaneous Velocity**

Suppose the position of a particle is given by the equation \( s(t) = -t^2 + 4t \).

a) Where is \( s(t) \geq 0 \)? Give your answer in interval notation.

   \( s(t) \) is a **quadratic**, thus graph is a **parabola**.
   Since leading coefficient is negative, graph opens down.

   Roots of \( s(t) = -t^2 + 4t \) are \( t = 0 \) and \( t = 4 \).
   \( s(t) \geq 0 \) in \( [0, 4] \).

b) Sketch the graph of \( s(t) \) over the interval you found in part a.)

   ![Graph of s(t)](image)(c) On your graph, draw dots at the points corresponding to \( t = 1 \) and \( t = 3 \), and sketch the secant line connecting them.

d) Formally calculate the average velocity over the interval \( [1, 3] \). Why could you have guessed this answer just from the shape of the graph, without doing any calculations? (Hint: think symmetry.)

   \[
   V_{avg} = \frac{s(3) - s(1)}{3 - 1} = \frac{[-9 + 12] - [-1 + 4]}{2} = \frac{3 - 3}{2} = 0.
   \]

   We could have guessed this because \( x = 1 \) and \( x = 3 \) are symmetric within the interval \( [0, 4] \).

e) Suppose you wanted to calculate the instantaneous velocity at \( t = 1 \). Describe in detail how you could use a table to calculate the answer. Give concrete examples of numbers and/or intervals you might use, and explain what needs to be calculated (though you need not calculate it.)

<table>
<thead>
<tr>
<th>Interval</th>
<th>( V_{avg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, 2]</td>
<td>( \frac{s(2) - s(1)}{2 - 1} )</td>
</tr>
<tr>
<td>[1, 1.1]</td>
<td>( \frac{s(1.1) - s(1)}{1.1 - 1} )</td>
</tr>
<tr>
<td>[1.1, 1]</td>
<td>( \frac{s(1) - s(1.1)}{1 - 1.1} )</td>
</tr>
</tbody>
</table>
Problem 4  LIMIT MISCELLANEA

a) Describe in your own words what the expression \( \lim_{x \to c} f(x) = L \) means. (Your description can be informal, but should be complete and accurate.)

As \( x \) gets arbitrarily close to \( c \), \( f(x) \) gets arbitrarily close to \( L \). [We exclude the case \( x = c \); we define \( f \) as a limit.]

b) Suppose that \( \lim_{x \to 0} f(x) = 3 \). Find \( \lim_{x \to 0} \frac{\sqrt{x} - 1}{x + 1} f(x) \).

\[
\lim_{x \to 0} \frac{\sqrt{x} - 1}{x + 1} f(x) = \left( \lim_{x \to 0} \frac{\sqrt{x} - 1}{x + 1} \right) \cdot \lim_{x \to 0} f(x) = \frac{0}{0} \cdot 3 = \frac{6}{6} = 1.
\]

\[\text{Diagram of } f(x) \text{ satisfying conditions.}\]

(c) Sketch the graph a function \( f(x) \) that satisfies the following:
\( \lim_{x \to 2^-} f(x) = -\infty \), \( \lim_{x \to 2^+} f(x) = \infty \), \( \lim_{x \to 4^-} f(x) = 1 \), \( \lim_{x \to 4^+} f(x) = 2 \).

\[\text{Sketch of graph.}\]

d) Concept question: suppose you know that \( f(3) = 2 \). Can \( \lim_{x \to 3} f(x) = 4 \)? Briefly explain why or why not.

Yes. The value \( f(3) = 2 \) at \( x = 3 \) has nothing to do with \( \lim_{x \to 3} f(x) \).

\[\text{Sketch of graph with value at } x = 3.\]

e) True or False: if for some \( f \) both the left hand limit \( \lim_{x \to c^-} f(x) \) exists and the right hand limit \( \lim_{x \to c^+} f(x) \) exists, then the two-sided limit \( \lim_{x \to c} f(x) \) must exist as well. Explain why or why not.

False. The left and right hand limits could be different, in which case \( \lim_{x \to c} f(x) \) would not exist.