Solutions

4.9.4

\[
\int (9x + 15x^{-2}) \, dx = 9 \int x \, dx + 15 \int x^{-2} \, dx \\
= 9 \cdot \frac{1}{2} x^2 + 15 \cdot \frac{x^{-1}}{-1} + C \\
= \frac{9}{2} x^2 - 15x^{-1} + C.
\]

As a check, we have

\[
\frac{d}{dx} \left( \frac{9}{2} x^2 - 15x^{-1} + C \right) = 9x + 15x^{-2}
\]
as needed.

4.9.8

\[
\int (e^x - 4 \sin x) \, dx = e^x - 4 \int \sin x \, dx \\
= e^x - 4(-\cos x) + C = e^x + 4\cos x + C.
\]

As a check, we have

\[
\frac{d}{dx} (e^x + 4\cos x + C) = e^x - 4\sin x
\]
as needed.

4.9.16

\[
\int 14s^{9/5} \, ds = 14 \cdot \frac{s^{14/5}}{14/5} + C = 5s^{14/5} + C.
\]

4.9.28

\[
\int (\theta + \sec^2 \theta) \, d\theta = \frac{1}{2} \theta^2 + \tan \theta + C.
\]

4.9.40 Let \( F(x) \) be an antiderivative of \( f(x) \). By definition, this means \( F'(x) = f(x) \). In other words, \( f(x) \) provides information as to the increasing/decreasing behavior of \( F(x) \). Since, moving left to right, \( f(x) \) transitions from \(-\) to \(+\) to \(-\) to \(+\) to \(-\) to \(+\), it follows that \( F(x) \) must transition from decreasing to increasing to decreasing to increasing to decreasing to increasing. This describes the graph in (A)!

4.9.48 Since \( \frac{dy}{dx} = 3 - 2t \), we have

\[
y = \int (3 - 2t) \, dt = 3t - t^2 + C.
\]

Thus,

\[
-5 = y(0) = 3(0) - (0)^2 + C = C,
\]

so that \( C = -5 \). Therefore, \( y = 3t - t^2 - 5 \).

5.1.7 Let \( f(x) = 2x + 3 \) on \([0,3]\).

(a) We partition \([0,3]\) into 6 equally-spaced subintervals. The left endpoints of the subintervals are \( \{0, \frac{1}{3}, 1, \frac{5}{3}, 2, \frac{7}{3} \} \) whereas the right endpoints are \( \{\frac{1}{2}, 1, \frac{5}{2}, 2, \frac{7}{2}, 3 \} \).

- Let \( a = 0, b = 3, n = 6, \Delta x = (b - a)/n = \frac{1}{2} \), and \( x_k = a + k\Delta x, k = 0, 1, \ldots, 5 \) (left endpoints).

Then

\[
L_6 = \sum_{k=0}^{5} f(x_k) \Delta x = \Delta x \sum_{k=0}^{5} f(x_k) = \frac{1}{2} (3 + 4 + 5 + 6 + 7 + 8) = 16.5.
\]
• With \( x_k = a + k\Delta x, k = 1, 2, \ldots, 6 \) (right endpoints), we have

\[
R_6 = \sum_{k=1}^{6} f(x_k) \Delta x = \Delta x \sum_{k=1}^{6} f(x_k) = \frac{1}{2} (4 + 5 + 6 + 7 + 8 + 9) = 19.5.
\]

(b) Via geometry (see figure below), the exact area is \( A = \frac{1}{2} (3)(6) + 3^2 = 18 \). Thus, \( L_6 \) underestimates the true area \( (L_6 - A = -1.5) \), while \( R_6 \) overestimates the true area \( (R_6 - A = +1.5) \).

![Graph](image.png)

5.1.16 Let \( f(x) = x^2 + x \) on \([-1, 1]\). For \( n = 5 \), \( \Delta x = (1 - (-1))/5 = \frac{2}{5} \), and \( \{x_k\}_{k=0}^{5} = \{-1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\} \). Therefore

\[
R_5 = \frac{2}{5} \sum_{k=1}^{5} (x_k^2 + x_k) = \frac{2}{5} \left( \left( \frac{9}{25} - \frac{3}{5} \right) + \left( \frac{1}{25} - \frac{1}{5} \right) + \left( \frac{1}{25} + \frac{1}{5} \right) + \left( \frac{9}{25} + \frac{3}{5} \right) + 2 \right)
\]

\[
= \frac{2}{5} \left( \frac{14}{5} \right) = \frac{28}{25}.
\]

5.1.20 Let \( f(x) = \ln x \) on \([1, 3]\). For \( n = 5 \), \( \Delta x = (3 - 1)/5 = \frac{2}{5} \), and \( \{x_k\}_{k=0}^{4} = \{\frac{6}{5}, \frac{8}{5}, 2, \frac{12}{5}, \frac{14}{5}\} \). Therefore,

\[
M_5 = \frac{2}{5} \sum_{k=0}^{4} \ln x_k
\]

\[
= \frac{2}{5} \left( \ln \frac{6}{5} + \ln \frac{8}{5} + \ln 2 + \ln \frac{12}{5} + \ln \frac{14}{5} \right) \approx 1.300224.
\]

5.1.22 The first term is \( 2^2 + 2 \), and the last term is \( 5^2 + 5 \), so it seems that the sum limits are 2 and 5, and the \( k \)th term is \( k^2 + k \). Therefore, the sum is:

\[
\sum_{k=2}^{5} (k^2 + k).
\]

5.1.28 (a) \( \sum_{j=3}^{4} \sin \left( \frac{j\pi}{2} \right) = \sin \left( \frac{3\pi}{2} \right) + \sin \left( \frac{4\pi}{2} \right) = -1 + 0 = -1. \)

(b) \( \sum_{k=3}^{5} \frac{1}{k - 1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}. \)

(c) \( \sum_{j=0}^{2} 3j - 1 = \frac{1}{3} + 1 + 3 = \frac{13}{3}. \)

5.1.46 Let \( f(x) = 3x + 6 \) on \([1, 4]\). Let \( N \) be a positive integer and set \( a = 1, b = 4, \) and \( \Delta x = (b - a)/N = (4 - 1)/N = 3/N. \) Also, let \( x_k = a + k\Delta x = 1 + 3k/N, k = 1, 2, \ldots, N \) be the right endpoints of the \( N \) subintervals of \([1, 4]\). Then

\[
R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{3}{N} \sum_{k=1}^{N} \left( 9 + \frac{9k}{N} \right)
\]

\[
= \frac{27}{N} \sum_{k=1}^{N} 1 + \frac{27}{N^2} \sum_{k=1}^{N} j = \frac{27}{N}(N) + \frac{27}{N^2} \left( \frac{N^2}{2} + \frac{N}{2} \right)
\]

\[
= \frac{81}{2} + \frac{27}{2N}.
\]

The area under the graph is

\[
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( \frac{81}{2} + \frac{27}{2N} \right) = \frac{81}{2}.
\]

The region under the graph is a trapezoid with base width 3 and heights 9 and 18. The area of the region is then \( \frac{1}{2}(3)(9 + 18) = \frac{81}{2}, \) which agrees with the value obtained from the limit of the right-endpoint approximations.
5.2.6 The region bounded by the graph of \( y = \sin x \) and the \( x \)-axis over the interval \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \) consists of two parts of equal area, one above the axis and the other below the axis. Hence,
\[
\int_{\pi/2}^{3\pi/2} \sin x \, dx = 0.
\]

5.2.14 Let \( f(x) \) be given by Figure 14.

(a) The definite integral \( \int_1^4 f(x) \, dx \) is the signed area of one-quarter of a circle of radius 1 which lies below the \( x \)-axis and one-quarter of a circle of radius 2 which lies above the \( x \)-axis. Therefore,
\[
\int_1^4 f(x) \, dx = \frac{1}{4} \pi (2)^2 - \frac{1}{4} \pi (1)^2 = \frac{3}{4} \pi.
\]

(b) The definite integral \( \int_1^6 |f(x)| \, dx \) is the signed area of one-quarter of a circle of radius 1 and a semicircle of radius 2, both of which lie above the \( x \)-axis. Therefore,
\[
\int_1^6 |f(x)| \, dx = \frac{1}{2} \pi (2)^2 - \frac{1}{4} \pi (1)^2 = \frac{9\pi}{4}.
\]

5.2.16 To make the value of \( \int_0^a g(t) \, dt \) as large as possible, we want to include as much positive area as possible. This happens when we take \( a = 4 \). Now, to make the value of \( \int_0^b g(t) \, dt \) as large as possible, we want to make sure to include all of the positive area and only the positive area. This happens when we take \( b = 1 \) and \( c = 4 \).

5.2.20 Let \( f(x) = 2x + 3 \). With
\[
P = \{x_0 = -4, x_1 = -1, x_2 = 1, x_3 = 4, x_4 = 8\} \quad \text{and} \quad C = \{c_1 = -3, c_2 = 0, c_3 = 2, c_4 = 5\},
\]
we get
\[
R(f, P, C) = \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) + \Delta x_4 f(c_4)
\]
\[
= (1 - (-4))(-3) + (1 - (-1))(3) + (4 - 1)(7) + (8 - 4)(13) = 70.
\]

Here is a sketch of the graph of \( f \) and the rectangles.

5.2.44 \( \int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right) \, dx = 2 \int_0^5 f(x) \, dx - \frac{1}{3} \int_0^5 g(x) \, dx = 2(5) - \frac{1}{3}(12) = 6. \)

5.2.56 \( \int_1^2 f(x) \, dx = \int_0^2 f(x) \, dx - \int_0^1 f(x) \, dx = 4 - 1 = 3. \)

5.2.60 \( \int_2^4 f(x) \, dx - \int_4^6 f(x) \, dx = \left(\int_2^4 f(x) \, dx + \int_4^6 f(x) \, dx\right) - \int_4^6 f(x) \, dx = \int_2^4 f(x) \, dx. \)

5.3.12 \( \int_{-1}^1 (5u^4 + u^2 - u) \, du = \left(u^5 + \frac{1}{3}u^3 - \frac{1}{2}u^2\right) \bigg|_{-1}^{1} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) - \left(-1 - \frac{1}{3} - \frac{1}{2}\right) = \frac{8}{3}. \)

5.3.20 \( \int_{-2}^{-1} \frac{1}{x^3} \, dx = -\frac{1}{2}x^{-2} \bigg|_{-2}^{-1} = -\frac{1}{2}(-1)^{-2} + \frac{1}{2}(-2)^{-2} = -\frac{3}{8}. \)
5.3.28 \( \int_{\pi/4}^{5\pi/8} \cos 2x \, dx = \frac{1}{2} \sin 2x \bigg|_{\pi/4}^{5\pi/8} = \frac{1}{2} \sin \frac{5\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} = -\frac{\sqrt{2}}{4} - \frac{1}{2} \). 

5.3.36 \( \int_2^3 e^{4t-3} \, dt = \frac{1}{4} e^{4t-3} \bigg|_{2}^{3} = \frac{1}{4} e^{9} - \frac{1}{4} e^{5} \).

5.3.40 \( \int_1^4 \frac{dt}{5t+4} = \frac{1}{5} \ln |5t+4| \bigg|_{1}^{4} = \frac{1}{5} \ln 24 - \frac{1}{5} \ln 9 = \frac{1}{5} \ln \frac{24}{9} \).

5.4.4 By definition, \( F(0) = \int_0^{\sqrt{2} + t} dt = 0 \). By FTC, \( F'(x) = \sqrt{x^2 + x} \), so that \( F'(0) = \sqrt{0^2 + 0} = 0 \) and \( F'(3) = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3} \).

5.4.8 \( F(x) = \int_2^x (12t^2 - 8t) \, dt = (4t^3 - 4t^2) \bigg|_{2}^{x} = 4x^3 - 4x^2 - 16 \).

5.4.21 By FTC II, \( \frac{d}{dx} \int_0^x (t^5 - 9t^3) \, dt = x^5 - 9x^3 \).

5.4.26 The graph of \( y = g(x) \) lies above the x-axis over the interval [0, 1], below the x-axis over [1, 3], and above the x-axis over [3, 4]. The corresponding area function should therefore be increasing on (0, 1), decreasing on (1, 3) and increasing on (3, 4). Further, it appears from Figure 9 that the local minimum of the area function at \( x = 3 \) should be negative. One possible graph of the area function is the following.

5.4.30 By the Chain Rule and the FTC, \( \frac{d}{dx} \int_1^{1/x} \cos^3 t \, dt = \cos^3 \left( \frac{1}{x} \right) \cdot \left( -\frac{1}{x^2} \right) = -\frac{1}{x^2} \cos^3 \left( \frac{1}{x} \right) \).

5.4.32 Let

\[
F(x) = \int_{x^2}^{x^4} \sqrt{t} \, dt = \int_{x^2}^{x^4} \sqrt{t} \, dt - \int_{0}^{x^2} \sqrt{t} \, dt.
\]

Applying the Chain Rule combined with FTC, we have

\[
F'(x) = \sqrt{x^3} \cdot 4x^3 - \sqrt{x^2} \cdot 2x = 4x^5 - 2x |x|.
\]