1. Modeling Problem #11

**Variables:**

Let $x_0$ denote the number of DVDs produced and $x_1$ denote the number of cell phones produced (both in thousands.)

**Constraints:**

Assets consist of cash, money owed from past sales, and the cash value of raw materials. Liabilities consist of debt.

**Assets:**

- **Cash assets:**
  \[
  \text{cash assets} = 100 \text{ (cash on hand)} - 10 \text{ (loan payment)} - 12 \text{ (rent)} - 65x_0 \text{ (money spent to produce DVDs)} - 45x_1 \text{ (money spend to produce phones)} + 20 \text{ (debt collected)}
  \]

- **Debt assets:**
  \[
  \text{debt assets} = 150x_0 + 100x_1 \text{ (money owed from sales this year)} + 35 \text{ (money owed from past sales)} - 20 \text{ (money owed from past sales that is payed off)}
  \]

- **Material assets:**
  \[
  \text{material assets} = 50 \text{ (current value of materials)} - 40x_0 - 20x_1 \text{ (material spent on production this year)} + 20 \text{ (shipment of materials received)}
  \]

**Liabilities:**

- **Debt liabilities:**
  \[
  \text{debt liabilities} = 100 \text{ (current debt)} - 10 \text{ (debt payed off)} + 20 \text{ (debt incurred by new shipment of materials.)}
  \]

The requirement that assets be twice liabilities at the year’s end is modeled as:

\[
\text{cash assets + debt assets + materials assets} \geq 2 \cdot \text{debt liability}
\]

where each term is as given above. Substitution each equation yields and taking all terms to the left hand side yields:

\[
183 + 45x_0 + 35x_1 - 220 \geq 0.
\]
The requirement that cash assets need to be 200000 is modeled as:

$$53 - 65x_0 - 45x_1 \geq 0.$$  

**Objective:**

The objective to be minimized is the negative of the total profit:

$$\Gamma(x_0, x_1) = -(45x_0 + 35x_1)$$  

**Solution:**

I solved this problem in Python, and my code is posted online. The (continuous) solution was

$$x_0 = 469.23 \quad x_1 = 500$$

2. Modeling Problem #13

**PART A**

**Variables:**

Let $$x_0, x_1, \ldots, x_5$$ denote the gallons of each ingredient devoted to the production of regular gas, and $$x_6, \ldots, x_{11}$$ the number of galls of each ingredient devoted to the production of premium.

**Constraints:**

Total caps on the daily amount of each ingredient available translate into six constraints:

$$x_0 + x_6 \leq 16113$$  
$$x_1 + x_7 \leq 14505$$  
$$x_2 + x_8 \leq 7083$$  
$$x_3 + x_9 \leq 2430$$  
$$x_4 + x_{10} \leq 576$$  
$$x_5 + x_{11} \leq 68452$$

The constraint that the second ingredient can comprise no more than 10% of the total mass of each kind of gasoline translates into two constraints:

$$x_1 \leq 0.1 \cdot (x_0 + \cdots + x_5)$$  
$$x_7 \leq 0.1 \cdot (x_6 + \cdots + x_{11})$$

I calculate the RVP of each kind of gas as the weighted average of the RVP of the ingredients. In particular, the RVP of the regular is

$$RVP_{reg} = \frac{3.22x_0 + 3.37x_1 + 11.43x_2 + 5.12x_3 + 4.97x_4 + 57.3x_5}{x_0 + \cdots + x_5}$$

and the RVP of the premium is

$$RVP_{prem} = \frac{3.22x_6 + 3.37x_7 + 11.43x_8 + 5.12x_9 + 4.97x_{10} + 57.3x_{11}}{x_6 + \cdots + x_{11}}$$
The upper bounds on RVP result in two constraints:

\[
\begin{align*}
3.22x_0 + 3.37x_1 + 11.43x_2 + 5.12x_3 + 4.97x_4 + 57.3x_5 & \leq 7.8 \cdot (x_0 + \cdots + x_5) \\
3.22x_6 + 3.37x_7 + 11.43x_8 + 5.12x_9 + 4.97x_{10} + 57.3x_{11} & \leq 7.8 \cdot (x_6 + \cdots + x_{11})
\end{align*}
\]

Similarly, I calculated the AKI as the weighted average of the AKI of the individual ingredients, where I got the AKI of the individual ingredients by taking the direct average of the RON and the MON. In particular, the AKI of the regular is

\[
AKI_{reg} = \frac{95.25x_0 + 90.65x_1 + 84.05x_2 + 94.1x_3 + 112.05x_4 + 95.35x_5}{x_0 + \cdots + x_5}
\]

and the AKI of the premium is

\[
AKI_{prem} = \frac{95.25x_6 + 90.65x_7 + 84.05x_8 + 94.1x_9 + 112.05x_{10} + 95.35x_{11}}{x_6 + \cdots + x_{11}}
\]

The upper and lower AKI bounds on regular, and the lower bound on premium, result in three constraints:

\[
\begin{align*}
95.25x_0 + 90.65x_1 + 84.05x_2 + 94.1x_3 + 112.05x_4 + 95.35x_5 & \leq 90 \cdot (x_0 + \cdots + x_5) \\
95.25x_0 + 90.65x_1 + 84.05x_2 + 94.1x_3 + 112.05x_4 + 95.35x_5 & \geq 87 \cdot (x_0 + \cdots + x_5) \\
95.25x_6 + 90.65x_7 + 84.05x_8 + 94.1x_9 + 112.05x_{10} + 95.35x_{11} & \geq 91 \cdot (x_6 + \cdots + x_{11})
\end{align*}
\]

Finally, I assume that expected demand must be met, i.e. we need to produce at least 7600 of regular and 2100 gallons of premium. This yields two more constraints:

\[
\begin{align*}
x_0 + \cdots + x_5 & \geq 7.6 \\
x_6 + \cdots + x_{11} & \geq 2.1
\end{align*}
\]

Objective:
The goal is to maximize revenue, i.e. to minimize

\[
\Gamma = -[3.60 \cdot (x_0 + \cdots + x_5) + 4.00 \cdot (x_6 + \cdots + x_{11})].
\]

Solution:
I solved this problem in Python, and my code is posted online. See the online script for solution details.

PART B
Part B was almost exactly the same, with a few shifts in numbers. Check the iPython solutions for exact details.

3. Modeling Problem #15

Variables
I used four variables, \(x_0, \cdots, x_3\), where \(x_i\) is the number of violins produced in the \(i + 1\)st quarter.

Constraints:
Demand must be met, so we need total number of violins on hand each quarter to be at least 5, 4, 3, and 3, respectively. Total number of violins on hand is the carryover from the previous quarter plus the number produced. I will assume that the “predicted demand” is the exact demand. This results in 4 equations:

\[
\begin{align*}
6 + x_1 &\geq 5 \\
(6 + x_1 - 5) + x_2 &\geq 4 \\
(6 + x_1 - 5 + x_2 - 4) + x_3 &\geq 3 \\
(6 + x_1 - 5 + x_2 - 4 + x_3 - 3) + x_4 &\geq 3
\end{align*}
\]

(In my Python code I wrote this as a single vector constraint.)

**Objective:**
The objective is to minimize cost. Each violin has a base cost of $200, and within each quarter, every violin after the second carries a surcharge of $100. Production costs are thus

\[\text{production costs: } = 200 \cdot \sum_0^3 x_i + 100 \cdot \sum_0^3 \max\{x_i - 2, 0\}.\]

Carrying costs are only incurred with excess stock. We can track them through each quarter as follows:

\[
\text{number of violins carried: } = 6 + x_0 - 5 \text{ (1st quarter)} + (6 + x_0 - 5) + x_1 - 4 \text{ (2nd quarter)} + (6 + x_0 - 5 + x_1 - 4) + x_3 - 3 \text{ (3rd quarter)} + (6 + x_0 - 5 + x_1 - 4 + x_3 - 3) + x_4 - 3 \text{ (3rd quarter)}
\]

Total carrying costs is thus $25 times this quantity. In my Python code I wrote this as a single expression involving a lower diagonal matrix, but it would be more straightforward to write it out directly.

**Solution:**
I solved this in Python, and got (2,2,2,3) as my solution.