1. Gradient Descent in 2-Dimensions

Suppose you are trying to find the minimizer via a gradient descent $f(x, y) = 2x^2 + y^2$, and your initial guess is the point (1, 1).

(a) Calculate $\nabla f(1, 1)$, the gradient of $f$ at the point (1, 1).

(b) Suppose you take a gradient descent step with no backtracking whatsoever. What is your next guess for the minimizer? Is $f$ at this guess larger, or smaller, than it is at (1, 1)?

(c) Now suppose instead you take a gradient descent step with “dumb” backtracking, as discussed in class. What is your next guess for the minimizer?

(d) Finally, suppose you take a gradient descent step with an exact line search. What is your next guess for the minimizer?
2. Rates of Convergence

(a) Write down what it means for a sequence \( x_n \) to converge to a number \( x^* \) \textbf{linearly}.

(b) Show that the sequence

\[ x_n = 1 + \left( \frac{1}{3} \right)^n \]

converges linearly to 1.

(c) Write down what it means for a sequence \( x_n \) to converge to a number \( x^* \) \textbf{quadratically}.

(d) Sketch the proof that Newton’s method converges quadratically. (Hint: remember that the proof starts with a Taylor series of the form \( f'(x) = f'(x_0) + f''(x_0)(x - x_0) + 0.5 \cdot f'''(\xi)(x - x_0)^2 \))
3. Constraints

Suppose \( f(x) = (x - 4)^2 \) and that \( x \) is constrained to the interval \([0, 2]\).

(a) What is the true value of the minimizer of \( f \) over \([0, 2]\)?

(b) Express the constraint \( x \in [0, 2] \) as two functional constraints of the form \( g_1(x) \geq 0 \) and \( g_2(x) \geq 0 \). What are \( g_1 \) and \( g_2 \)?

(c) Let \( \tilde{f}(x) = f(x) - \min\{g_1(x)^3, 0\} - \min\{g_2(x)^3, 0\} \). Sketch the graph of \( \tilde{f} \) and put a dot at its minimizer. Is this the true minimizer of the original problem?

(d) Suppose you modified \( \tilde{f} \) to be of the form \( \tilde{f}(x) = f(x) - \lambda_1 \cdot \min\{g_1(x)^3, 0\} - \lambda_2 \cdot \min\{g_2(x)^3, 0\} \), where \( \lambda_1 \) and \( \lambda_2 \) are positive multipliers. If you wanted to drive the penalized solution towards the true solution of the original problem, how would you adjust \( \lambda_1 \) and/or \( \lambda_2 \)?
4. Simplex Method

Suppose your goal is to minimize a linear objective of the form $f(x, y) = -x + 4y$ subject to the constraints $y \geq 2x - 1$, $y \leq 1$, and $y \geq -x - 1$.

(a) Write each of the constraints in the form $g_i(x, y) \geq 0$, where $g_1$ corresponds to the first constraint above, $g_2$ the second, and $g_3$ the third.

(b) Write down $\nabla f$ and $\nabla g_i$ for $i = 1, 2, 3$.

(c) Sketch the constraint region. Include a few level lines of $f$, and at each vertex, sketch and label the gradient vectors of the adjacent constraint functions.

(d) Find $\lambda_1$ and $\lambda_2$ such that $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, and use your results, together with the multiplier rule, to explain why the simplex method (i.e. the working set method) would not halt at the point $(1, 1)$. 
5. **Duality**

Consider the same linear program as in the previous problem, i.e. the problem of minimizing a linear objective of the form $f(x, y) = -x + 4y$ subject to the constraints $y \geq 2x - 1$, $y \leq 1$, and $y \geq -x - 1$.

(a) Write this problem in matrix form, i.e. identify a matrix $A$ and vectors $c$ and $b$ such that the problem becomes

$$
\min_x c^T \cdot x \quad \text{such that} \quad Ax \geq b.
$$

(b) Write down the dual problem.

(c) Verify that $(2, 1, 3)$ satisfies the constraints of the dual problem.

(d) What lower bound on the original problem can you derive from the triple $(2, 1, 3)$?