1. Newton’s Method in one dimension.

(a) Suppose you are using Newton’s Method to solve for a root of \( f(x) = x^2 - 4x - 7 \), and your initial guess is \( x_0 = 3 \). Without doing any math, draw an appropriate tangent line, and use it to identify the position of the next Newton guess. Label this point \( x_1 \). Is \( x_1 \) the exact root?

(b) Formally solve for \( x_1 \).

(c) Now suppose you are using Newton’s Method to find the minimizer of \( f \), and once again your initial guess is \( x_0 = 3 \). Formally solve for the next Newton guess. Draw a dot to indicate its location on the graph, and label the dot \( x'_1 \).

(d) Superimpose a sketch \( f'(x) \) on the graph above, and use this sketch to explain, in a single sentence, why Newton’s method works better for finding maxs/mins than roots of quadratics.
2. Modeling.

Suppose you need 3 units of caffeine and 3 units of sugar to make it through the morning. At Bob’s Java Hut you can order the house coffee for $2.00/liter, or the house soda for $1.00/liter. Each liter of coffee contains 3 units of caffeine and 1 unit of sugar, while each liter of soda provides 1 unit of caffeine and 3 units of sugar.

Write and solve an optimization model to answer the following question: what is the least expensive way to get your morning fix if you wish to meet the requirements exactly? Include the following:

- a sketch of the feasible region (i.e. the coffee-soda combinations that meet the constraints) and a dot indicating where your solution lies within this region
- a list of any assumptions you are making
- a single sentence comment about how the price of coffee relative to the price of soda influences the result in this problem.
3. **Taylor Series in one dimension.** Let \( f(x) = e^{2x} + x \).

(a) Find the second order Taylor approximation of \( f(x) \) at the point \( x_0 = 0 \).

(b) Use this Taylor series to approximate \( f(1) \).

(c) What does the Remainder Theorem say about the error of this approximation? Give a formula (hint: it should involve a mystery point.)

(d) Without solving for the mystery point, use your answer in part c to provide an upper bound on the magnitude of the error.
4. **Stopping.**

Recall that if we are using Newton’s Method to find a root $\pi$ of a function in one dimension, the “improved stopping criterion” we use is of the form

$$\text{STOP IF } |f(x)| < L\epsilon,$$

where $L$ is the “scaling factor” and $x$ is our current guess.

(a) How is $L$ defined, in theory?

(b) Explain in your own words how this stopping criterion guarantees that $|x - \pi| < \epsilon$. (Suggestion: use the Mean Value Theorem and your definition of $L$.)

(c) How do we estimate $L$ in practice? Why is this estimation necessary, and how is it justified? (A couple of key words suffice for the latter.)

(d) What is the stopping criterion in $n$-dimensions?
5. **Newton’s Method for two dimensional optimization.** Let \( f(x, y) = 2x^2 + 5y^2 - 2xy \).

(a) Find the gradient of \( f(x, y) \) at the point \((1, 1)\).

(b) Find the Hessian of \( f(x, y) \) at the point \((1, 1)\).

(c) Find the inverse of the Hessian at \((1, 1)\).

(d) If \((1, 1)\) is your initial guess for the minimizer of \( f(x, y) \), and you’re using Newton’s method, what is your next guess?