Homework 2

Theory:

1. Consider a function \( f(x) = x^3 \) with domain \( \mathbb{R}^+ \). Suppose you have a strange computer which can store only odd integers. Your computer approximates \( f \) with a different function, \( \tilde{f}(x) \), defined as

\[
\tilde{f}(x) = \tilde{x}^3,
\]

where \( \tilde{x} \) is the closest odd integer to \( x \) (rounding up if \( x \) happens to be an even integer.) Find strict upper and lower bounds for the condition number of the problem of finding \( f(x) \) with this computer for \( x \) in the range \([2, 4]\).

2. Show that if \( x \in \mathbb{R}^n \) satisfies the least squares problem

\[
\min_x \| Ax - b \|^2, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m,
\]

then \( x \) satisfies the normal equations, i.e.

\[
A^T Ax = A^T b.
\]

(Here, the superscript \( T \) stands for transpose.)

Computation:

For computation problems, turn in a coversheet that has the solutions, but that does not contain the code used to generate those solutions. As with the theory problems, the solutions should be couched in complete sentences and fleshed out as necessary. Separately, turn in whatever matlab functions you used to calculate these solutions.

1. Suppose you are a scientist working on immune response, and you develop a model of the following form:

\[
w = \frac{a2^b + 100 \cos(x/50)}{\sqrt{c}},
\]

where

- \( w \) = white blood cell count
- \( a, b, c \) = levels of certain essential minerals in the bloodstream
- \( x \) = age of the patient, in years

Suppose a 60 year old patient is tested for her levels of \( a, b, \) and \( c \) and finds them to be:

\[
a = 10 \\
b = 5 \\
c = 1
\]

Do the following:

(a) Calculate her expected white blood cell count.

(b) Suppose the values for \( a, b, \) and \( c \) are know up to about 1%. Estimate the range of values that \( w \) could assume under changes of this magnitude. (Hint: do this numerically, i.e. generate a bunch of random perturbations of the right size, recalculate \( w \), and keep track of the results.)

(c) Form a plot with three subplots (hint: use the \texttt{subplot(1,3)} command in Matlab.) The first plot should show how \( w \) changes as \( a \) changes from \( 0.99 \cdot a \) to \( 1.01 \cdot a \). The second should show how \( w \) changes as \( b \) changes from \( 0.99 \cdot b \) to \( 1.01 \cdot b \), and the third should show how \( w \) changes as \( c \) changes from \( 0.99 \cdot c \) to \( 1.01 \cdot c \).

(d) Comment on which of the three factors the white cell count is most sensitive to.