Exam 2

Please endeavor to write neatly, show your work, and circle your answers. This test is closed note, closed book, and closed calculator.

1 Areas and Volumes

Instructions: Set up the integrals that give the requested volumes in the following problems. YOU DO NOT NEED TO SOLVE THE INTEGRALS.

1. (5 points) For $x$ in the range $[-1, 1]$, find the area of the region between the graphs of $f(x) = x^3 - 4x$ and $g(x) = x$.

   \[
   A = \int_{-1}^{1} [f(x) - g(x)] \, dx
   \]

2. (5 points) For $x$ in the region $[0, 4]$, let $R$ be the region below the graph of $f(x) = x^2 + 1$. Find the volume of the solid whose base is $R$ and whose cross-sections perpendicular to the $x-$axis are squares.

   \[
   V = \int_{0}^{4} x \cdot 2 \, dx
   \]

3. (5 points) Use the method of washers to calculate the volume of the solid generated by revolving the region below $y = (x - 1)^2$, $0 \leq x \leq 1$, around the $x-$axis.

   \[
   V = \pi \int_{0}^{1} [(x - 1)^2 - 0^2] \, dx
   \]
4. (5 points) Use the method of cylindrical shells to calculate the volume of the solid generated by revolving the region below \( y = -x(x-1) \) for \( 0 \leq x \leq 1 \) around the \( y \)-axis.

\[
V = 2\pi \int_0^1 x (-x)(x-1) \, dx
\]

5. (5 points) Suppose that you know \( f(x) \) is a positive function, but you don't know what it is. Consider the region between the graph of \( f(x) \) and the \( x \)-axis, for \( 0 \leq x \leq 2 \). Write an integral that gives the volume of the solid obtained by revolving this region around the line \( x = 3 \). Are you using the method of washers or the method of shells?

\[
V = 2\pi \int_0^2 (3-x) f(x) \, dx
\]

6. (5 points) Suppose that you know \( f(x) \) is a positive function, but again you don't know what it is. Consider the region between the graph of \( f(x) \) and the \( x \)-axis, for \( 0 \leq x \leq 2 \). Write an integral that gives the volume of the solid obtained by revolving this region around the line \( y = -3 \). Are you using the method of washers or the method of shells?

\[
V = \pi \int_0^2 \left[ f(x) + 3 \right]^2 - (0 + 3)^2 \, dx
\]

7. (5 points) **Social mixer:** If you prefer using the method of washers, draw a shell. If you prefer using the method of shells, draw a washer. Does drawing the Enemy help mitigate your anxiety?
2 Techniques of Integration

*Instructions:* The following reduction formulas may be of use in some of the following problems:

\[
\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx
\]

\[
\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx
\]

For each of the following, evaluate the definite or indefinite integral.

1. (5 points) \( \int xe^{-2x} \, dx \)

\[
\begin{align*}
\text{Let } u &= x, & u' &= 1 \\
\text{Let } v &= e^{-2x}, & v' &= -2e^{-2x} \\
\end{align*}
\]

\[
= -xe^{-2x} + \int e^{-2x} \, dx = -xe^{-2x} - \frac{1}{2} e^{-2x} + C
\]

2. (5 points) \( \int \sin^2 x \cos^3 x \, dx \)

\[
\begin{align*}
\int \sin^2 x \cos x \, dx &= \int \cos x \cos \sin^2 x \, dx \\
&= \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C
\end{align*}
\]

3. (5 points) \( \int \sqrt{1-x^2} \, dx \)

\[
\begin{align*}
\text{Let } x &= \sin \theta, & dx &= \cos \theta \, d\theta \\
\int \sqrt{1-x^2} \, dx &= \int \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta \\
&= \int \cos \theta \, d\theta = \frac{1}{2} \theta + C = \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x + C
\end{align*}
\]

4. (5 points) \( \int \frac{1}{(x-5)(x-2)} \, dx \)

\[
\begin{align*}
\int \frac{A}{x-5} + \frac{B}{x-2} \, dx &= A \ln |x-5| + B \ln |x-2| = -\frac{1}{3} (\ln |x-5| - \frac{1}{3} \ln |x-2|) + C
\end{align*}
\]

For \( A, B > 0 \):

\[
\begin{align*}
\frac{A}{x-5} + \frac{B}{x-2} &= \frac{A}{x-5} + \frac{B}{x-2} \\
A(x-2) + B(x-5) &= 1 \\
A(2) + B(-3) &= 1 \Rightarrow \begin{cases} A = \frac{1}{7} \Rightarrow B = \frac{5}{7} \end{cases}
\end{align*}
\]
5. (5 points) \[ \int e^x \cos x \, dx = \int e^{x} \cos x \, dx \]

\[ u = e^x \quad u' = e^x \quad v = \cos x \quad v' = -\sin x \]

\[ w = e^x \quad w' = e^x \quad z = \sin x \quad z' = \cos x \]

\[ \Rightarrow \int e^x \cos x \, dx = \int e^x \left[ \cos x - \frac{1}{2} \sin x \right] \, dx + C \]

6. (5 points) \[ \int \sin^2 x \cos^3 x \, dx \]

\[ \int e^x \cos x \, dx = \int e^x \left[ \cos x - \frac{1}{2} \sin x \right] \, dx + C \]

7. (5 points) \[ \int \frac{1}{\sqrt{2x^2 + 4}} \, dx \]

8. (5 points) \[ \int \frac{1}{x^2 + 1} \, dx \]

9. (5 points) Once you finish this problem, you will have completed 80 out of 100 points. 10 of these points were veritable freebies. Draw an icon that represents how this makes you feel.
3 Miscellaneous

1. (10 points) **True or False:** For each of the following statements, mark T if the statement is necessarily true (i.e. always true in all circumstances), otherwise mark F. **IMPORTANT:** If you mark F, either give a counter example or explain your reasoning.

   - \( \text{T (F)} \quad \csc(\pi/3) = \sqrt{3}/2 \)

   - \( \text{T (F)} \quad \csc(x) = \frac{2}{\sin x} \)

   - \( \text{T (F)} \quad \) If you generate a solid by revolving a region around the \( y \)-axis, and wish to use the method of cylindrical shells to find the solid's volume, then you will integrate along the \( x \)-axis.

   - \( \text{T (F)} \quad \) Let \( f(x) \) be a function defined on the interval \([0, 2]\), and consider the region bound between the graph of \( f(x) \) and the \( x \)-axis. For any such \( f(x) \), the volume of the solid obtained by revolving this region around the \( y \)-axis is given by \( V = 2\pi \int_0^2 xf(x) \, dx \).

   - \( \text{T (F)} \quad \) The derivative of \( \csc x \) is \( -\cot x \csc x \).

   - \( \text{T (F)} \quad \) Let \( f(x) \) be a function defined on the interval \([0, 2]\), and consider the region bound between the graph of \( f(x) \) and the \( x \)-axis. For any such \( f(x) \), the volume of the solid obtained by revolving this region around the \( x \)-axis is given by \( V = \pi \int_0^2 [f(x)]^2 \, dx \).

2. (10 points) Use what we’ve learned about how to find volumes of solids of revolution to derive the formula for the volume of a sphere of radius \( r \). (Hint: draw a semi-circle of radius \( r \) positioned along the \( x \)-axis and centered at the origin. The equation for this semi-circle is \( y = \sqrt{r^2 - x^2} \). Now revolve the semi-circle around the \( x \)-axis.)

\[
V = \frac{\pi}{3} \int \left( \sqrt{r^2 - x^2} \right)^2 \, dx
\]

\[
= \frac{\pi}{3} \int \left( r^2 - x^2 \right) \, dx
\]

\[
= \frac{\pi}{3} \int \left( r^2 - x^2 \right)^{3/2} \, dx
\]

\[
= \frac{2}{3} \pi \left[ \frac{2}{3} \right] = \frac{2}{3} \pi \cdot \frac{2r^3}{3} - \frac{4}{3} r^3
\]