Exam 1

Please endeavor to write neatly, show your work, and circle your answers. This test is closed note, closed book, and closed calculator.

(1) (25 Points) Consider the following definite integral:
\[ \int_{-1}^{2} 3x \, dx. \]

(a) Evaluate the integral geometrically. (I.e. draw a picture, and use this, together with elementary geometry, to figure out the value of the integral.)

[Diagram showing geometric interpretation with points A, B, and C labeled.]

\[ A = \frac{1}{2} \cdot 1 \cdot 3 = -\frac{3}{2} \]
\[ B = \frac{1}{2} \cdot 2 \cdot 6 = 6 \]
\[ A + B = 6 - \frac{3}{2} = \frac{9}{2} = 4.5 \]

(b) Now approximate the integral via a Riemann sum with partition \( P = \{-1, 1/2, 3/2, 2\} \) and sample points \( C = \{0, 1, 2\} \).

\[ \Delta x_1 = \frac{1}{2}, \quad \Delta x_2 = 1, \quad \Delta x_3 = \frac{1}{2} \]
\[ f(x_1) = 0, \quad f(x_2) = 3, \quad f(x_3) = 6 \]
\[ R(f, P, C) = \left| \frac{1}{2} \cdot 0 + 1 \cdot 3 + \frac{1}{2} \cdot 6 \right| \]
\[ = 3 + 3 = 6 \]
\[ R(f, P, C) = \]

(c) What is the norm, \( \|P\| \), of the partition you used in part 1b) above?

\[ \text{largest} \Delta x_i = \frac{1}{2} = \|P\|. \]
(d) Now suppose you want to approximate the integral using n rectangles of equal widths, using the right hand endpoints of these rectangles as your sample points. Write a formula for your approximation.

\[ \Delta x = \frac{2 - (-1)}{n} = \frac{3}{n} \]

\[ c_i = -1 + i \cdot \Delta x = -1 + \frac{3i}{n} \]

\[ f(c_i) = 3\left(-1 + \frac{3i}{n}\right) = -3 + \frac{9i}{n} \]

\[ A = \sum_{i=1}^{n} \Delta x \cdot f(c_i) = \sum_{i=1}^{n} \frac{3}{n} \cdot \left(-3 + \frac{9i}{n}\right) = \sum_{i=1}^{n} -\frac{9}{n} + \sum_{i=1}^{n} \frac{27i}{n^2} \]

\[ R_n = \frac{-9 \cdot n + \frac{27}{n^2} \cdot n(n+1)}{2} \]

\[ R_n = \frac{-9 + \frac{27}{2} \cdot \frac{n(n+1)}{n}}{n} \]

(e) Evaluate your formula for \( n = 3 \) and \( n = 9 \).

\[ R_3 = \frac{-9 + \frac{27}{2} \cdot \frac{4}{3}}{3} = -9 + 9 \cdot \frac{2}{3} = -\frac{9}{3} + 6 = \frac{10}{3} \]

\[ R_9 = \frac{-9 + \frac{27}{2} \cdot \frac{10}{9}}{9} = -9 + 3 \cdot \frac{10}{9} = \frac{-90 + 30}{9} = -\frac{60}{9} = -\frac{20}{3} \]

(f) Now evaluate \( R_n \) in the limit as \( n \to \infty \) (show your work!)

\[ \lim_{n \to \infty} R_n = \left[ -9 + \frac{27}{2} \cdot \frac{n(n+1)}{n} \right] = -9 + \frac{27}{2} = -\frac{18}{2} + 13.5 = 4.5 \]

\[ \lim_{n \to \infty} R_n = \frac{-9 + \frac{27}{2} \cdot \frac{n(n+1)}{n}}{n} \]

(g) If you did the last part correctly, the answer that you got should be the correct value of the integral. Referring back to 1e), which approximation was better, \( R_3 \) or \( R_9 \)? Why might this be the case?

\[ R_9 \text{ (used more rectangles.)} \]
(2) (10 points) State the two versions of the Fundamental Theorem of Calculus. For each theorem statement, give an example of the theorem in action. (i.e. use the theorem to solve a concrete problem of your invention.)

**Version 1:**

**Theorem:**
$$\int_a^b f(x) \, dx = F(b) - F(a), \quad \text{with} \quad \frac{d}{dx} F(x) = f(x).$$

**Example:**
$$\int_1^2 x \, dx = \left. \frac{x^2}{2} \right|_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}.$$ 

**Version 2:**

**Theorem:**
$$\frac{d}{dx} \int_a^x f(c) \, dc = f(x), \quad \text{with} \quad c = \text{constant}.$$ 

**Example:**
$$\frac{d}{dx} \int_2^x \cos \theta \, d\theta = \cos x.$$ 

(3) (3 points) Ah, three pages down. Three more to go. Slow and steady wins the race, of course: sketch the Calculus Tortoise in his Calculating Clothes.
(4) (25 points) Evaluate the following definite integrals:

(a) \[ \int_{-2}^{2} (x^3 - 2x + 3) \, dx \]

\[ \frac{x^4}{4} - \frac{x^2}{2} + 3x \bigg|_{-2}^{2} = \left[ \frac{2^4}{4} - \frac{2^2}{2} + 3 \cdot 2 \right] - \left[ \frac{(-2)^4}{4} - \frac{(-2)^2}{2} + 3 \cdot (-2) \right] = 4 - 4 + 6 - [4 - 4 - 6] = \boxed{12} \]

(b) \[ \int_{-2}^{2} \frac{2}{x^2} \, dx \]

\[ = -2x^{-1} \bigg|_{-2}^{-2} = -2 \left[ \frac{1}{-2} - \frac{1}{-1} \right] = 2 \left[ \frac{2 - 1}{2} \right] = \boxed{1} \]

(c) \[ \int_{0}^{\pi} \sin(4x) \, dx \]

\[ = -\frac{\cos(4x)}{4} \bigg|_{0}^{\pi} = -\frac{\cos(4\pi)}{4} - \left[ -\frac{\cos(0)}{4} \right] = -\frac{1}{4} + \frac{1}{4} = \boxed{0} \]

(d) \[ \int_{3}^{5} e^{-4x+1} \, dx \]

\[ = e \int_{3}^{5} e^{-4x} \, dx = e \cdot \frac{e^{-4x}}{-4} \bigg|_{3}^{5} = e \cdot \frac{e^{-20}}{-4} - e \cdot \frac{e^{-12}}{-4} = \boxed{2} \]

(e) \[ \int_{-2}^{1} \frac{1}{x} \, dx \]

\[ = \ln |x| \bigg|_{-2}^{1} = \ln 1 - \ln (-2) = \boxed{-\ln 2} \]
(5) (25 points) Evaluate the following integrals (note that some are definite, some are indefinite.)

(a) \[ \int \frac{dx}{\sqrt{5 - 3x^2}} = \int \frac{du}{\sqrt{1 - \frac{3}{5} u^2}} \quad \text{with} \quad u = \frac{\sqrt{5}}{3} x \]
\[ c = \frac{5}{\sqrt{3}} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{\sqrt{3}} \int \frac{dv}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{3}} \sin^{-1} \left( \frac{\sqrt{5}}{3} x \right) \]

(b) \[ \int_0^1 3^{-x} dx = \left. -3^{-x} \right|_{0}^{1} = \frac{-1}{\ln 3} + \frac{1}{\ln 3} = \frac{2}{\ln 3} \]

(c) \[ \int \cos^5 \theta \sin \theta d\theta \quad \text{let} \quad u = \cos \theta \]
\[ dv = -\sin \theta \ d\theta \]
\[ \int \cos^5 \theta \sin \theta d\theta = -\frac{1}{6} = -\frac{\cos^6 \theta}{6} \]

(d) \[ \int_{\pi/4}^{\pi/2} \theta \cot(\theta^2) d\theta \quad \text{let} \quad u = \theta^2 \]
\[ dv = 2 \theta d\theta \quad u(\pi/4) = \frac{\pi^2}{16} \quad u(\pi/2) = \frac{\pi^2}{4} \]
\[ \int_{\pi/4}^{\pi/2} \theta \cot(\theta^2) d\theta = \frac{1}{2} \int_{\pi/16}^{\pi^2/4} \frac{\cos u}{\sin u} \ du = \frac{1}{2} \int_{\pi/16}^{\pi^2/4} \frac{\cos u}{\sin u} \ du = \frac{1}{2} \left[ \ln(\sin u) \right]_{\pi/16}^{\pi^2/4} \]

(e) \[ \int_{-1}^{1} \sqrt{5x + 6} dx \quad \text{let} \quad u = \frac{x}{5} + \frac{6}{5} \]
\[ du = \frac{1}{5} dx \]
\[ = \frac{1}{5} \int_0^{2} \sqrt{u} \ du = \frac{1}{5} \left[ \frac{2}{3} u^{3/2} \right]_0^{16} = \frac{1}{5} \cdot \frac{2}{3} \cdot 16 = \frac{16}{15} = 0.63 \]
(6) (10 points) True or False: For each of the following statements, mark T if the statement is necessarily true (i.e., always true in all circumstances), otherwise mark F. IMPORTANT: If you mark F, either give a counter example or explain your reasoning.

T F Let \( R_n \) represent the Riemann sum approximation of some integral using \( n \) rectangles of equal width and right hand endpoint. Then if \( n > m \), \( R_n \) is a better approximation of the integral than \( R_m \).

T F The Fundamental Theorem of Calculus is useful for solving both definite and indefinite integrals.

\[ \text{Both versions have } \int_a^b \text{ with definite integrals.} \]

T F For any differentiable functions \( g(x) \) and \( f(x) \) and any constant \( c \),

\[ \frac{d}{dx} \int_c^{g(x)} f(u) \, du = f(g(x))g'(x). \]

T F Let \( f \) be an increasing function, and consider the integral \( \int_a^b f(x) \, dx \). If \( R_n \) the Riemann sum approximation of the integral using \( n \) equi-spaced rectangles and right hand endpoints, then \( R_n \) will overestimate the integral.

T F \( \sec(\pi/3) = \sqrt{3}/2. \)

(7) (2 Points) Free at last! Free at last! Derive an equation for happiness.