Exam 2

(1) **Areas:** (10 points) Write down integrals that give the areas of the regions described below. You do not need to evaluate the integrals. (Note: graphs are sketched on Figure 1 on the last page.)

(a) The area between the curves $f(x) = x^2$ and $g(x) = x$, for $1 \leq x \leq 3$.

$$
\int_{1}^{3} (x^2 - x) \, dx
$$

(b) The area between the same curves, for $0 \leq x \leq 3$.

$$
\int_{0}^{1} (x^2 - x) \, dx + \int_{1}^{3} (x^2 - x) \, dx
$$

(2) **General Volumes:** (10 points) Write down integrals that give the volumes of the regions described below. You do not need to evaluate the integrals. (Graphs sketched in Fig. 2.)

(a) The base of a solid is the region between the curve $y = x^2$ and the interval $[0, 2]$ on the $x-$axis. The cross sections perpendicular to the $x-$axis are squares.

$$
\int_{0}^{2} A(x) \, dx = \int_{0}^{2} (x^2 - x^2) \, dx
$$

(b) The same solid, only the cross sections perpendicular to the $x-$axis are equilateral triangles.

$$
\int_{a}^{b} A(x) \, dx = \int_{0}^{2} \frac{x^4}{2\sqrt{3}} \, dx \quad \text{or} \quad \int_{0}^{2} \frac{x^2}{2} \, dx
$$

![Two interpretations]
(3) **Volumes of Revolution: (25 points)** Write down integrals that give the volumes of the regions described below. You do not need to evaluate the integrals. (Graphs sketched in Fig. 3.)

(a) The volume generated by rotating the area under \( f(x) = \sin(x) \) for \( 0 \leq x \leq \pi \) around the \( x \)-axis.

\[
\pi \int_0^\pi \sin^2 x \, dx
\]

(b) The volume generated by rotating the same area around the \( y \)-axis.

\[
\pi \int_0^\pi 2\pi x \sin x \, dx
\]

(c) The volume generated by rotating the same area around around the line \( y = -2 \)

\[
\pi \int_0^\pi \left[ (\sin x + 2)^2 - 2 \right] \, dx
\]

(d) The volume generated by rotating the same area around the line \( y = 2 \).

\[
\pi \int_0^\pi \left( 2 - \left[ 2 - \sin x \right]^2 \right) \, dx
\]

(e) The volume generated by rotating the same area around the line \( x = -3 \).

\[
2\pi \int_0^\pi (x+3) \sin x \, dx
\]
(4) **Lengths:** (10 points) Write down integrals that give the lengths of the regions described below. You do not need to evaluate the integrals.

(a) The length of the curve \( y = \tan x \) for \( 0 \leq x \leq \pi/4 \).

\[
L = \int_{\alpha}^{\beta} \sqrt{1 + (y')^2} \, dx = \int_{\alpha}^{\beta} \sqrt{1 + \sec^2 x} \, dx
\]

(b) The length of the curve given parametrically by \( x = \sin t, \ y = \cos t \), for \( 0 \leq t \leq \pi \).

\[
L = \int_{a}^{b} \sqrt{(x')^2 + (y')^2} \, dt = \int_{a}^{b} \sqrt{\cos^2 t + \sin^2 t} \, dt = \int_{a}^{b} 1 \, dt
\]

(5) **Surface Areas:** (10 points) Write down integrals that give the surface areas of the regions described below. You do not need to evaluate the integrals.

(a) The surface swept out by revolving the line \( y = x^2 + 1 \) for \( 1 \leq x \leq 4 \) around the \( x \)-axis.

\[
S = \int_{1}^{4} 2\pi x \sqrt{1 + (2x)^2} \, dx
\]

(b) The surface swept out by revolving the same line around the \( y \)-axis.\[
\frac{\partial x}{\partial y} = \frac{1}{2(y-1)^2} \quad \text{if } y \neq 1
\]

\[
x = 1 \Rightarrow y = 2
\]

\[
y = 1 \Rightarrow y = 17
\]
(6) **Differential Equations:** (5 points) Let $T(t)$ denote the temperature of a cup of coffee at time $t$, and $R$ denote the temperature of the room in which it is sitting. (Note that $R$ does not change with time.) Suppose that the rate at which $T$ changes is proportional to the difference between the temperature of the coffee and the room temperature. Write down a differential equation that expresses the time evolution of $T$.

$$\frac{dT}{dt} = h(T - R)$$

(7) **Art Break (5 points):** Draw an abstract representation of your pending spring break. Include at least one circus animal.
(8) Integration: (25 points)

(a) \( \int x \sin x \, dx \)
\[ u = x \quad dv = \sin x \, dx \]
\[ du = dx \quad v = -\cos x \]
\[ = -x \cos x + \int \cos x \, dx \]
\[ = -x \cos x + \sin x \]

(b) \( \int \sin^4 x \, dx \)
\[ \frac{1 - \cos 2x}{2} \]
\[ \cos 2x = \frac{1 + \cos 2x}{2} \]
\[ = \int \frac{1}{2} - \cos 2x + \frac{1 + \cos 2x}{8} \, dx \]
\[ = \frac{1}{2} x - \frac{\sin 2x}{2} + \frac{1}{8} x + \frac{\sin 2x}{16} \]

(c) \( \int e^x \cos x \, dx \)
\[ u = e^x \quad dv = \cos x \, dx \]
\[ du = e^x \, dx \quad v = \sin x \]
\[ = e^x \sin x - \int e^x \sin x \, dx \]
\[ \int e^x \sin x \, dx \]
\[ = e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x \, dx \right] \]

(d) \( \int \sqrt{1 + \cos x} \, dx \)
\[ 2 \cos \frac{x}{2} = 2 \cos \left( \frac{x}{2} \right) - 1 \]
\[ = \int \sqrt{\frac{1}{2} + 2 \cos \left( \frac{x}{2} \right) - 1} \, dx \]
\[ = 2 \int \cos \left( \frac{x}{2} \right) \, dx \]

(e) \( \int \sec^4 x \, dx \)
\[ = \int \sec^2 x \, dx \cdot \sec^2 x \, dx \]
\[ = \int (1 + \tan^2 x) \sec^2 x \, dx \]
\[ = \frac{\sec^2 x + \tan x}{3} \]
Figures

Figure 1. Problem 1

Figure 2. Problem 2

Figure 3. Problem 3