Test 1

Please justify your answers. It is fine to leave answers in an unsimplified form, as long as this form is correct.

**Problem 1** How many different 5-place codewords are possible if the first and last place are for numbers and the middle three places are for letters? (Assume 26 letters, i.e. do not differentiate upper from lower case.)

$$
\begin{aligned}
&\text{Basic Rule of Counting} \\
&26 \cdot 10
\end{aligned}
$$

**Problem 2** Gene A is present in 20% of all people, gene B in 80%. If 10% of all people carry both genes, what percentage of people carry at least one of them?

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
A &= \text{carries gene } A \text{ or } B \text{ or both} \\
B &= \text{carries gene } B
\end{aligned}
$$

$$
\begin{aligned}
P(\text{at least one}) &= P(A \cup B) \\
&= .2 + .8 - .1 \\
&= .9
\end{aligned}
$$

**Problem 3** A committee of 2 vegetarians, 2 rastafarians, and 3 dangerously hard rockers is to be formed from a group containing 4 vegetarians, 5 rastafarians, and 5 dangerously hard rockers. How many committees are possible?

$$
\begin{aligned}
\binom{4}{2} \binom{5}{2} \binom{5}{3}
\end{aligned}
$$

Basic Rule of Counting

1st experiment: choose two vegetarians

2nd experiment: choose two rastafarians

3rd experimental: choose 3 dangerously hard rockers
Problem 4  What is the probability that a hand of 5 cards will contain one or more pairs?

\[ P(A) = P(\text{contains at least one pair}) = P(A) - P(\text{all cards are different}) \]

\[ P(A) = \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \cdot \frac{36}{48} \]

So, cases:

\[ 1 - \frac{48 \cdot 44 \cdot 40 \cdot 36}{51 \cdot 50 \cdot 49 \cdot 48} = \frac{2}{5} \]

Problem 5  A person has 10 friends, 6 of whom will be invited to go fishing. How many choices are there if 2 of the friends will only attend together?

Two cases:

1. Both attend. (In which case 4 slots remain)

\( \binom{8}{4} \) ways of filling these slots

2. Neither attends. (In which case 6 slots remain)

\( \binom{6}{4} \) ways of filling these slots.

Ans. \( \binom{8}{4} + \binom{6}{4} \)

Problem 6  If 12 people are to divide themselves into three groups of sizes 3, 3, and 6, how many unique groupings are possible?

Once two groups of 3 have been chosen, the remaining group is chosen by default (since \( 3 + 3 + 6 = 12 \)). So, how many ways are there of choosing 2 groups of 3 from the 6 people that do not form the last group?

Ans. \( \binom{6}{2} \)

Since there are \( \binom{12}{6} \) ways of choosing those people, the total of 3 groupings is:

\[ \frac{12!}{2! \cdot 3! \cdot 6!} \]

Problem 7  Two cards are randomly selected from a deck. What is the probability that they form a blackjack (i.e. that one is an ace and the other is a ten, a jack, a queen, or a king?)

2 cases

1. Draw ace 1st, then the "10" card.

\[ \frac{4}{52} \cdot \frac{16}{51} \]

2. Draw "10" card 1st, then the ace card.

\[ \frac{16}{52} \cdot \frac{4}{51} \]

Since 1 & 2 are disjoint, probabilities sum.

Ans. \( \frac{2 \cdot 4 \cdot 16}{52 \cdot 51} \)
Problem 8  An urn contains 4 green, 3 white, and 5 blue balls. What is the probability that a set of three randomly chosen balls will be of the same color?

\[A = \text{all green}\]
\[B = \text{all white}\]
\[C = \text{all blue}\]

\[P(A \cup B \cup C) = P(A) + P(B) + P(C)\]

\[= \frac{4}{12} \cdot \frac{3}{11} \cdot \frac{2}{10} + \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{1}{10} + \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \]

Problem 9  Repeat the last problem under the assumption that after every draw, the color of the ball is noted and the ball is placed back in the urn.

\[D = \text{all } A, B, \text{ or } C\]

\[P(A) = \left(\frac{4}{12}\right)^3\]
\[P(B) = \left(\frac{3}{12}\right)^3\]
\[P(C) = \left(\frac{5}{12}\right)^3\]

\[P(A \cup B \cup C) = P(A) + P(B) + P(C)\]

\[= \frac{4^3 + 3^3 + 5^3}{12^3} \]

Problem 10  If a five-sided die is rolled five times in succession, what is the probability that a five appears at least once?

\[A = \text{five appears at least once}\]

\[P(A) = 1 - P(A')\]

\[= 1 - P(3 \text{ no 5 appears})\]

\[= 1 - \left(\frac{4}{5}\right)^5\]