Quiz: Chapter 6

Unless otherwise marked, each problem is worth 7 points. Please endeavor to show enough work that I can divine your thought process: this will facilitate assigning partial credit.

Problem 1  The joint density function of \( X \) and \( Y \) is given by

\[
f(x,y) = \begin{cases} 
2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\
0 & \text{otherwise}
\end{cases}
\]

Compute \( P\{X > 1, Y < 1\} \)

\[
\int_1^\infty \int_0^{\frac{\exp(x)}{2}} 2e^{-x}e^{-2y} \, dy \, dx = \int_1^\infty 2e^{-x} \left( \frac{1}{2} - \frac{e^{-x}}{2} \right) \, dx = \int_1^\infty -e^{-x} \left[ e^{-x} - \frac{1}{2} \right] \, dx
\]

\[
= \left( 1 - e^{-x} \right) \cdot e^{-x} \Bigg|_1^\infty = 1 - \frac{1}{2} = \frac{1}{2}.
\]

Problem 2  A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 pm, find the probability that the first to arrive has to wait longer than 10 minutes.

\[
x = \text{man's arrival time} \\
y = \text{woman's arrival time} \\
\mathbb{1}(x) = \begin{cases} 
1 & \text{on } [0,1] \\
0 & \text{otherwise}
\end{cases}
\]

\[
P(x < y) = \int_0^1 \int_0^y 1 \, dx \, dy = \int_0^1 y \, dy = \frac{1}{2} [y^2]_0^1 = \frac{1}{2}.
\]

Need: \( P(x < y - 10) = P(y < x + 10) \).

By symmetry, suffice to find one & have no duplicate it:

\[
2P(x < y - 10) = 2 \int_0^{y-10} \int_0^1 dx \, dy = 2 \int_0^{y-10} \left( \frac{y-10}{2} \right) \, dy = 2 \left[ \frac{y^2}{2} - \frac{(y-10)^2}{2} \right]_{10}^{y}
\]

\[
= 2 \left[ \frac{y^2}{2} - \frac{(y-10)^2}{2} \right]_{10}^{y} = 2 \left[ \frac{y^2}{2} - \frac{1}{2} \right] = \frac{y}{6} + \frac{2}{72} = \frac{25}{36}.
\]
**Problem 3** Suppose $X$ and $Y$ are independent and uniformly distributed on $[0, 1]$. Find the probability density of $X + Y$. 

\[ f_{XY}(x,y) = \begin{cases} 1 & x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \]

Two cases:

- Case 1: \( x \leq 1 \)
  \[ P(X+Y \leq z) = \frac{z}{2} \]

- Case 2: \( z > 1 \)
  \[ P(X+Y \leq z) = \frac{z}{2} + \left( \frac{1}{2} - \frac{z-1}{2} \right) \]
  \[ = 1 - \frac{(z-1)^2}{2} \]
  \[ = 1 - \frac{z^2 + 1}{2} + \frac{z}{2} \]

\[ S_2: \begin{cases} z < 1 \\ \frac{z}{2} \end{cases} \]

\[ 1 - 2z + \frac{z^2}{2} \]

\[ f_X(z) = \begin{cases} \frac{z}{2} & z \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

**Problem 4** Suppose that $p(x,y)$, the joint probability mass function of $X$ and $Y$, is given by

\[ p(0,0) = 0.4, \quad p(0,1) = 0.2, \quad p(1,0) = 0.1, \quad p(1,1) = 0.3. \]

Calculate the conditional probability mass function of $X$ given that $Y = 1$. 

\[
\begin{array}{c|cccc}
 x & 0 & 1 & 2 & 3 \\
 \hline
 0 & 0.4 & 0.1 & 0.3 & 0.1 \\
 1 & 0.2 & 0.4 & 0.1 & 0.2 \\
 \end{array}
\]

\[ P(X=0 | Y=1) = \frac{0.4}{0.5} = \frac{4}{5} \]

\[ P(X=1 | Y=1) = \frac{0.2}{0.5} = \frac{2}{5} \]

\[ P(X=2 | Y=1) = \frac{0.3}{0.5} = \frac{3}{5} \]
Problem 5  Suppose that the joint density of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{e^{-x^2/y} e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1 | Y = y\}$.

$$P_{X | Y}(x | y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{e^{-x^2/y} e^{-y}}{y}}{\frac{1}{Y}} = \frac{e^{-x^2/y} e^{-y}}{Y} \int_0^\infty \frac{e^{-x^2/y} e^{-y}}{y} \, dx = e^{-y} \int_0^\infty \frac{e^{-x^2}}{x} \, dx = -e^{-y}$$

$$\therefore P(X > 1 | Y = y) = \int_1^\infty P_{X | Y}(x | y) \, dx = \int_1^\infty \frac{e^{-x^2/y} e^{-y}}{Y} \, dx = -e^{-y} \bigg|_1^\infty = e^{-y}$$

Problem 6  Two fair dice are rolled. Find the joint probability density function of $X$ and $Y$ if $X$ is the value of the second die and $Y$ is the smaller of the two values.

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