Chapter 7 Outline: Expectation

September 22

Key Concepts

- For any random variables $X$ and $Y$ with joint density $f(x,y)$, the expectation of a function of $X$ and $Y$ is given by
  \[ E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)g(x,y). \]

- Consequently, expectation is a linear operator:
  \[ E\left[ \sum_{i=1}^{n} \alpha_i X_i \right] = \sum_{i=1}^{n} \alpha_i E[X_i] \]
  where the $X_i$ are random variables and the $\alpha_i$ are scalars.

- The covariance of two random variables $X$ and $Y$ is given by
  \[ \text{Cov}(X,Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y. \]
  Note that the covariance of a random variable with itself is just its variance.

- The correlation between two random variables is defined as
  \[ \rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \]

- Conditional expectations leverage conditional density functions (resp. prob. mass functions):
  \[ E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx, \]
  where
  \[ f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(x,y)} \]

- The moment generating function of a random variable $X$ is defined as
  \[ M(t) = E[e^{tX}]. \]
  The moment generating function can be difficult to compute, but once computed, it is a useful means of finding the moment of $X$ via the formula
  \[ E[X^n] = \frac{d^n}{dt^n}M(t)|_{t=0} \quad n = 1, 2, \ldots \]

- For any random variables $X_1, X_2, \ldots X_n$, we define the sample mean as
  \[ \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \]
  and the sample variance as
  \[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2. \]
  If the $X_i$ are normal, independent, and identically distributed $\mathcal{N}(\mu, \sigma)$, then the sample mean is also normal with mean $\mu$ and variance $\sigma^2/n$. Moreover, the random variable $(n - 1)S^2/\sigma^2$ is a chi-squared random variable with $n - 1$ degrees of freedom.