The Deterministic EOQ and EPQ with Partial Backordering at a Rate that Is Linearly Dependent on the Time to Delivery

Abstract

Our original models for the EOQ and EPQ with partial backordering assumed that the backordering rate, $\beta$, is a constant. In this paper we extend those models to allow $\beta$ to increase linearly as the time until delivery decreases. We show how those previous models can be adapted to find the optimal decision variable values for this new assumption and develop, for each model type, a condition that the initial value of $\beta$ must meet for partial backordering to be optimal.

Keywords: Inventory control; EOQ; Production lot sizing; Partial backordering

1. Introduction

Models for the basic economic order quantity with partial backordering (EOQ-PBO) were developed by Montgomery et al. (1973), Rosenberg (1979), Park (1982) (1983), Wee (1989), and Pentico and Drake (2009). Comparable models for the basic economic production quantity model with partial backordering (EPQ-PBO) were developed by Mak (1987), Zeng (2001), and Pentico et al. (2009). These are all single-item models that assume all parameters are known and constant over an infinite time horizon and, for the EPQ models, that replenishment is at a finite rate $P$. They assume the cost to place and receive an order is independent of the size of the order, the unit holding cost is an amount per unit per year, the unit backorder cost is an amount per unit per year plus, for Montgomery et al. (1973) and Rosenberg (1979), a fixed cost per unit backordered, and the lost sale cost is an amount per unit. With the exception of one of the models in Montgomery et al. (1973), they also all assume a constant backordering rate $\beta$. 
Pentico et al. (2010) extended their EPQ-PBO model with a constant $\beta$ to allow the value of $\beta$ to depend on the phase of the stockout interval, having a value $\beta_1 = \beta$ from the start of the stockout interval until production begins and a value $\beta_2 = \rho \beta$, $1 \leq \rho \leq 1/\beta$, from the start of production until the backorder is eliminated. If $\rho = 1$, then $\beta$ is constant during the entire stockout interval, as in Pentico et al. (2009). If $\rho = 1/\beta$, then partial backordering only applies during the initial phase of the stockout interval, as in Mak (1987) and Zeng (2001). If $1 < \rho < 1/\beta$, then the fraction of demand backordered once production starts increases, but partial backordering still applies. Pentico et al. (2010) suggested that this increase in $\beta$ might be due to the buyer’s reduced uncertainty about when delivery will take place since production has begun. They also said that the assumption that $\beta$ increases as the time to delivery gets shorter is consistent with much of the research on the economic order quantity model with partial backordering (EOQ-PBO) over the past 40 years.

Although Montgomery et al. (1973) focused primarily on the basic version of the problem in which $\beta$ is a constant, they noted that, while “a constant ratio…may be satisfactory in many practical situations,” in other cases it would be more appropriate to assume that the backordering percentage would increase as the time at which the replenishment order is to be received approaches. The only model for time-sensitive $\beta$ for which they presented any results is one in which $\beta$ increases linearly from its initial value, $\beta_0$, when the system first goes out of stock to 1.0 at the time when the replenishment order is delivered.

Many authors have developed models for the EOQ-PBO with other types of equations for time-sensitive $\beta$. San José et al. (2005a) (2005b) (2006) (2007) studied a variety of model forms, including linear, exponential, and rational. A number of other authors, including Abad (1996)
(2001) (2008) developed models for the EOQ-PBO that included, in addition to time-sensitive $\beta$, additional features, such as pricing, perishable inventory, or time- or stock-level-dependent demand. A comprehensive survey of this literature may be found in Pentico and Drake (2010).

While there have been many papers discussing the EOQ-PBO with time-sensitive $\beta$, almost all of which include other complicating features, most of the research on the EPQ-PBO assumes that $\beta$ is a constant, although several papers address the EPQ-PBO with additional features and time-sensitive $\beta$. Abad (2003) included pricing and perishable inventory in a model in which $\beta$ increases according to either an exponential or a rational function as the time to order fulfillment decreases, but he did not include the costs of backordering or lost sales in his profit function, which tends to bias the results toward higher stockouts. Skouri and Papachristos (2003) included production, demand, and inventory deterioration rates that changed with time. They assumed that $\beta$ increases according to a rational function of the time until production starts, after which, as in Mak (1987) and Zeng (2001), $\beta = 1.0$. Unlike Mak and Zeng, however, Skouri and Papachristos ignored the backordering cost once production begins. Zhou et al. (2003) included time-varying demand and deteriorating inventory over a finite planning horizon. They assumed the fraction of demand backordered is negatively linearly related to the current number of backorders, an approach that is the opposite of that taken by most other authors since it implies that the percent of stockouts backordered will decrease as the lead time decreases.

One thing that all the models using a continuous non-linear time-sensitive function for $\beta$ have in common is that they cannot be solved by simply substituting parameter values into closed-form expressions, which is the case for the basic EOQ-PBO model with linearly changing $\beta$ in Montgomery et al. (1973) and the EPQ-PBO with phase-dependent $\beta$ in Pentico et al.
These other models require either non-linear programming software or some sort of more complicated search procedures. The models we develop here are of the closed-form-expression type.

In this paper we will show how the models in Pentico and Drake (2009) and Pentico et al. (2009) can be modified to include \( \beta \) changing continuously as a linear function of the time remaining until delivery for both the EOQ-PBO and the EPQ-PBO. Our purposes in doing this are two-fold. The first is the obvious one of developing a model that may more accurately reflect the realities of the market. While we cannot cite field-based research that indicates that a time-sensitive function \( \beta(t) \) is a better representation of market reality than using a constant \( \beta \), the fact that the vast majority of research over the past 15 years uses this approach indicates that most researchers in this area do make that assumption. Having \( \beta \) change linearly with time is the simplest such assumption. The second, and more general, is to demonstrate how our approach to establishing the optimality of partial backordering extensions of the basic EOQ and EPQ models can be extended to more general situations with minimal analytic modification. Since the analysis is somewhat more complicated for the EPQ-PBO, we will develop the model for that case in Section 2 and then show in Section 3 how the results can be easily modified for the simpler EOQ-PBO.

2. **Model development for the EPQ-PBO with linearly changing \( \beta(t) \)**

Since our model is based on Pentico et al.’s (2009) model for the basic EPQ-PBO, we will use their notation with the addition of a new variable – \( L \), the total lost demand per year – and a change from a constant \( \beta \) to \( \beta(t) \), since the fraction of demand backordered will be a function of time. The notation is summarized in Table 1.
As in Pentico and Drake (2009) and Pentico et al. (2009) (2010), the decision variables are $T$, the time between orders or the length of an inventory cycle, and $F$, the fill rate or the percentage of demand filled from stock. Also, as in Pentico et al. (2009) (2010), we assume that once production starts, the existing backorders will be filled before the incoming demand, so some of that demand during the initial part of the production phase will be backordered and some will be lost sales.

2.1 The equation for $\beta(t)$

As in Montgomery et al. (1973), and referring to the graph of the inventory level over the course of an inventory cycle in Figure 1, we assume that $\beta(t)$, the fraction of demand that will be backordered at time $t$, starts at $\beta(0) = \beta_0$ at the start of a stockout interval and increases linearly with respect to time to $\beta((1 - F)T) = 1.0$ at the end of a stockout interval. That is:

$$\beta(t) = \beta_0 + (1 - \beta_0) \left( \frac{t}{(1 - F)T} \right) \text{ for } 0 \leq t \leq (1 - F)T.$$  \hspace{1cm} (1)

2.2 The objective function

The average cost per year is the sum of four terms: 1) the cost of placing and receiving orders, 2) the cost of holding inventory, 3) the cost of the stockouts that are backordered, and 4) the cost of the stockouts that are lost sales.

$$\Gamma(T,F) = \frac{C_o}{T} + C_h \bar{I} + C_b \bar{B} + C_d L.$$  \hspace{1cm} (2)

2.2.1 Average inventory

As shown in Figure 1, the in-stock interval of a production-inventory cycle, which has length $FT$, can be divided into two subintervals. Since $\beta(t)$ has no effect on the increase and
decrease in the positive inventory levels during the subintervals of length \( t_3 \) and \( t_4 \), the combined length of which is \( FT \), the equations for \( t_3 \) and \( t_4 \) are the same as they are in Pentico et al. (2009) (2010):

\[
t_3 = FTD/P \text{ and } t_4 = FT(1 - D/P)
\]

(3)

and the average inventory level is:

\[
\bar{I} = \frac{DTF^2(1 - D/P)}{2}
\]

(4)

2.2.2 Cumulative and average backorder levels

Again referring to Figure 1, the stockout interval, which is of length \( (1 - F)T \), can also be divided into two subintervals.

During the first phase of the stockout interval, which is of length \( t_1 \), there is no production, so the backorder level increases at a rate of \( \beta(t)D \) until it reaches its maximum level, \( B \). During the second phase of the stockout interval, which is of length \( t_2 \), production takes place at a rate of \( P \), which is higher than the rate at which the incoming demand is being backordered, so the backorder level changes at a rate of \( \beta(t)D - P \) until it reaches a level of zero at \( t = (1 - F)T \). Thus the rate at which the backorder level is changing at time \( t \) is:

\[
B'(t) = \begin{cases} 
\beta(t)D & 0 \leq t \leq t_1 \\
\beta(t)D - P & t_1 < t \leq (1 - F)T \\
0 & (1 - F)T < t
\end{cases}
\]

(5)

From this we get the equations for the cumulative backorder level at time \( t \):
To determine \( t_1 \), the length of the first phase of the stockout interval, set \( B((1 - F)T) = 0 \) and solve for \( t_1 \):

\[
B((1 - F)T) = \beta_0(1 - F)DT + \frac{(1 - \beta_0)(1 - F)DT}{2} - P((1 - F)T - t_1) = 0
\]

\[
\rightarrow t_1 = (1 - F)T \left[ 1 - \frac{D(\beta_0 + 1)}{2P} \right] = (1 - F)T \left[ 1 - \frac{\bar{\beta}D}{P} \right] \tag{7}
\]

where \( \bar{\beta} = (\beta_0 + 1)/2 \) is the average value of \( \beta(t) \) over the stockout interval. Then, since \( t_1 + t_2 = (1 - F)T \), we get:

\[
t_2 = (1 - F)T - t_1 = (1 - F)T(\bar{\beta} D/P). \tag{8}
\]

A graph of \( B(t) \), the backorder level at time \( t \), is the vertical mirror image of the left side of Figure 1 followed by \( B(t) = 0 \) for the rest of the inventory cycle, so \( \bar{B} \), the average backorder level is the average value of \( B(t) \) during the first part of the inventory cycle multiplied by \( (1 - F) \), the fraction of an inventory cycle during which there are backorders.
\[
\bar{B} = \left[ \frac{1}{(1-F)T} \int_0^{(1-F)T} B(t) dt \right] (1-F) = \frac{1}{T} \int_0^T B(t) dt
\]

\[
= \frac{D}{T} \int_0^T \left[ \beta_t + \left( \frac{(1-\beta(t))^2}{2(1-F)T} \right) \right] dt - \frac{P}{T} \int_{t_i}^{T} (t-t_i) dt
\]

\[
= \frac{D}{T} \left[ \frac{\beta_0 (1-F)^2 T^2}{2} + \frac{(1-\beta_0)(1-F)^2 T^2}{6} \right]
\]

\[
- \frac{P}{T} \left[ \frac{(1-F)^2 T^2}{2} - (1-F)^2 T^2 (1-\tfrac{\bar{\beta}D}{P}) + \frac{(1-F)^2 T^2}{2} (1-\tfrac{\bar{\beta}D}{P})^2 \right]
\]

\[
= \frac{D}{T} \left[ \frac{\beta_0 (1-F)^2 T^2}{2} + \frac{(1-\beta_0)(1-F)^2 T^2}{6} \right] - \frac{P}{T} \left[ \frac{(1-F)^2 T^2}{2} \left( \tfrac{\bar{\beta}D}{P} \right)^2 \right]
\]

\[
= D(1-F)^2 T \left[ \frac{\beta_0}{2} + \frac{(1-\beta_0)}{6} - \frac{(1+\beta_0)^2}{8P} \right] \tag{9}
\]

### 2.2.3 Lost sales

The total lost sales per cycle is the total demand per cycle during the stockout interval, \(D(1-F)T\), minus the total backordered demand per cycle, given by the integral of \(D\beta(t)\) over the range \(0 \leq t \leq (1-F)T\). Dividing this by \(T\) gives the average lost sales per year:

\[
L = \frac{[D(1-F)T - D \int_0^{(1-F)T} \beta(t) dt]}{T} = D(1-F) - \frac{D}{T} \int_0^{(1-F)T} \left[ \beta_0 + \frac{(1-\beta_0)t}{(1-F)T} \right] dt
\]

\[
= D(1-F) \frac{(1-\beta_0)}{2} = D(1-F)(1- \bar{\beta}) \tag{10}
\]

Note that either \((1-\beta_0)/2\) or \((1- \bar{\beta})\) makes sense as the multiplier in (10) since they are simply two ways of computing the average percentage of demand that results in lost sales when \(\beta(t)\) grows linearly over time during the stockout interval.

8
2.2.4 The complete objective function

Substituting the equations for $I$ from (4), $B$ from (9), and $L$ from (10) into (2), we get:

$$
\Gamma(T,F) = \frac{C_o}{T} + \frac{C_hDTF^2(1-D/P)}{2} + C_hD(1-F)^2\left[\frac{\beta_o}{2} + \frac{(1-\beta_o)}{6} - \frac{(1+\beta_o)^2D}{8P}\right] + C_lD(1-F)\frac{(1-\beta_o)}{2}.
$$

(11)

To simplify the notation, we define:

$$
C'_h = C_h(1-D/P)
$$

as was done in Pentico et al. (2009) (2010), and:

$$
C'_b = C_h\left[\beta_o + \frac{(1-\beta_o)}{3} - \frac{(1+\beta_o)^2D}{4P}\right].
$$

(13)

Then $\Gamma(T,F)$ can be rewritten as:

$$
\Gamma(T,F) = \frac{C_o}{T} + \frac{C_hDTF^2}{2} + \frac{C'_hDT(1-F)^2}{2} + C_lD(1-F)\frac{(1-\beta_o)}{2}.
$$

(14)

2.3 The values of T and F that minimize $\Gamma(T,F)$

In this section we develop explicit formulae for the values $T^*$ and $F^*$ that minimize the cost function $\Gamma$. We also establish necessary and sufficient conditions under which these formulae are valid.

With two exceptions, (14) here is identical to (14) in Pentico et al. (2009), the objective function in their model for the EPQ-PBO with constant $\beta$. The exceptions are: 1) There is no $\beta$ in the numerator of the third (backordering cost) term of (14) here because $\beta_0$ is built into the definition of $C'_h$, and 2) in the last term of (14) here, $(1-\beta)$ has been replaced by $(1-\beta_0)/2$,.
which is, as noted above, the average percentage of demand during the stockout interval that results in lost sales, as \((1 - \beta)\) is in the model in Pentico et al. (2009).

2.3.1 The equations for \(T^*\) and \(F^*(T)\)

Since (14) here is exactly the same as (14) in Pentico et al. (2009) if \(\beta C'_h DT(1 - F)^2/2\), the backordering cost, and \(C_i D(1 - F)(1 - \beta)\), the lost sales cost, in (14) in that paper are replaced by \(C'_h DT(1 - F)^2/2\) and \(C_i D(1 - F)(1 - \beta_0)/2\) respectively, the equations for \(T^*\) and \(F^*\) here are the same as those given in (19) and (18) of Pentico et al. (2009) if \(\beta C'_h\) and \(C_i(1 - \beta)\) in (19) and (18) in that paper are replaced by \(C'_h\) as defined in (13) and \(C_i(1 - \beta_0)/2\) respectively. This gives:

\[
T^* = \frac{2C_o}{DC'_h} \left[ \frac{C'_h + C'_b}{C'_h} \right] - \frac{[C_i(1 - \beta_0)/2]^2}{C'_h C'_b}
\]

\[(15)\]

\[
F^* = F^*(T^*) = \frac{C_i[(1 - \beta_0)/2] + C'_h T^*}{T^* (C'_h + C'_b)}
\]

2.3.2 Necessary and sufficient conditions for \(T^*\) and \(F^*\) to be optimal

Although (15) and (16) yield numerical values for \(T^*\) and \(F^*\) for any choice of problem parameters, not all such \(T^*\) and \(F^*\) actually represent optimal solutions. Pentico et al. (2009) showed that the equations for \(T^*\) and \(F^*\) yield optimal values only if the following condition on \(\beta\) is met:

\[
\beta \geq \beta^* = 1 - \sqrt{\frac{2C_o C'_h D}{(DC_i)}}.
\]

Note, moreover, that this condition is necessary but not sufficient. As pointed out by Zhang (2009) with respect to Pentico et al. (2009), to show sufficiency one also needs to check that the
total cost \( I(T^*, F^*) \) is less than \( C_iD \) (i.e. the cost of not stocking the item at all). We formalize this result as follows:

**Proposition:** The values of \( T^* \) and \( F^* \) computed from (15) and (16) give the minimum value for the objective function in (14) if and only if the following two conditions hold:

1) \( \beta_0 \geq \beta_0^* = 1 - 2\sqrt{2C_0C_hD / (DC_f)} \).  

2) \( I(T^*, F^*) \leq C_iD \).

The proof exactly mirrors that of Appendix B in Pentico et al. (2009), subject to the following definitional revisions:

\[
G_0 = C_o \quad G_1 = D(C_{h} + C_{b})/2 \quad G_2 = DC_{b}/2 \quad G_3 = C_iD(1 - \beta_0)/2.
\]  

The details are left to the reader.

Note that the square root in the numerator of (17) is the optimal average cost per year of using the basic EPQ without backordering; the denominator is the average cost per year of not stocking the item and having all lost sales. This is identical to the condition for optimality in Pentico et al. (2009) except that the square root in the numerator here is multiplied by 2, whereas there is no multiplier in the comparable equation for \( \beta^* \) in Pentico et al. (2009). Multiplying by 2 makes sense in this case, however, since, given that \( \beta(t) \) changes linearly, \( 1 - \beta_0 \) is twice the average fraction of lost sales.

### 2.3.3 The optimal value of \( I(T, F) \)

As suggested above, one way to determine the optimal value of \( I(T^*, F^*) \) is, of course, to simply substitute the values for \( T^* \) and \( F^* \) into the objective function expression in (14).

However, Pentico et al. ((2009), Eq. (20)) stated that:
\[ \Gamma^* = \Gamma(T^*, F^*) = C_h^1 DT^* F^*. \] (19)

It is straightforward to show that the same result applies here. They further noted that this same basic equation, with appropriate definition of \( C_h^1 \), applies to the basic EOQ, the EOQ with full backordering, the EOQ with partial backordering and a constant \( \beta \), the basic EPQ, the basic EPQ with full backordering, and the basic EPQ with partial backordering and a constant \( \beta \). In Pentico et al. (2010), they stated that (19) also applies to the EPQ with partial backordering with phase-dependent \( \beta \).

2.4 Process for determining the optimal solution

Given the results in the preceding subsections, the process for determining the values of \( T^* \), \( F^* \), and \( \Gamma^* \) is:

Step 1: Determine the values of \( C_h^1 \) and \( C_h^1 \) from (12) and (13) respectively.

Step 2: Determine the value of \( \beta_0^* \) from (17).

a. If \( \beta_0 < \beta_0^* \), do not backorder. Calculate \( T = \sqrt{2C_0 / DC_h^1} \), \( F = 1.0 \), and \( \Gamma(T,F) = C_h^1 DT\). If \( \Gamma(T,F) \leq C_iD \), set \( T^* = T \) and \( F^* = F \). If \( \Gamma(T,F) > C_iD \), do not stock the item. Stop.

b. If \( \beta_0 \geq \beta_0^* \), go to Step 3.

Step 3: Calculate \( T \) and \( F \) from (15) and (16) respectively. Calculate \( \Gamma(T,F) = C_h^1 DT\). If \( \Gamma(T,F) \leq C_iD \), set \( T^* = T \) and \( F^* = F \). If \( \Gamma(T,F) > C_iD \), do not stock the item. Stop.
3. Model Modification for the EOQ-PBO with Linearly Changing $\beta(t)$

Figure 2 shows the inventory level and time intervals for the EOQ-PBO with linearly increasing percentage of backordering. Comparing the graphs in Figures 1 and 2 is instructive. If the production rate in Figure 1 is increased, the graph looks basically the same except that the upward sloping lines in the intervals of length $t_2$ and $t_3$ – the two subintervals of the production interval – are steeper. The more the production rate is increased, the steeper those two lines become and, correspondingly, the shorter those two subintervals will be. If the production rate becomes orders of magnitude larger than the usage rate, those two lines will be almost vertical, their two intervals will almost disappear, and the modified version of Figure 1 will be virtually identical to the one in Figure 2, in which $t_1 = (1 - F)T, t_4 = FT$, and there are no intervals of length $t_2$ and $t_3$. That is, there is no production interval with subintervals during which, first, the backorders are eliminated and, second, inventory is gradually built up to its maximum level $I$.

The way in which this is usually expressed in stating the conditions for an EOQ model with or without backordering is that “replenishment is instantaneous.” Another way of expressing this idea, as Wee (1989) did, is to say that “the replenishment rate is infinite.” This highlights the idea that the only significant difference between the basic EPQ and EOQ without backordering models is that in the EPQ model replenishment is at a finite rate $P$ and in the EOQ the replenishment rate is $\infty$. The same idea holds for the EPQ and EOQ with backordering models, whether that backordering is full or partial.

Since, as just discussed, the EOQ-PBO model is the limit of the EPQ-PBO model as $P$ approaches $\infty$, the simplest way of determining the equations and optimality condition for the
EOQ-PBO, given that we have already determined the equations for the EPQ-PBO, is to set $P=\infty$ in those equations in Section 2.

Following this approach, we get the following:

1. Neither Equation (1) for $\beta(t)$ nor Equation (2) for $I(T,F)$ changes.

2. As noted, $t_3 = 0$ and $t_4 = FT$ (Equation (3)) and Equation (4) becomes:

$$\bar{I} = \frac{DTF^2}{2} \quad \text{(20)}$$

as it was in Pentico and Drake (2009).

3. As noted, $t_1 = (1 - F)T$ (Equation (7)) and $t_2 = 0$ (Equation (8)).

4. The transformed version of Equation (9) for $\bar{B}$ is:

$$\bar{B} = D(1 - F)^2 T \left[ \frac{\beta_0}{2} + \frac{(1 - \beta_0)}{6} \right] \quad \text{(21)}$$

4. Equation (10) for $L$ does not change.

5. The objective function (Equation (11)) becomes:

$$\Gamma(T,F) = \frac{C_a}{T} + \frac{C_h DTF^2}{2} + C_b D(1 - F)^2 T \left[ \frac{\beta_0}{2} + \frac{(1 - \beta_0)}{6} \right] + C_l D(1 - F) \left[ \frac{1 - \beta_0}{2} \right]. \quad \text{(22)}$$

6. As a result, $C_h$ is not replaced by $C_h'$ and $C_h' = C_b \left[ \beta_0 + \frac{(1 - \beta_0)}{3} \right]. \quad \text{(23)}$

7. The adjusted version of the objective function (Equation (14)) then becomes:

$$\Gamma(T,F) = \frac{C_a}{T} + \frac{C_h DTF^2}{2} + \frac{C_h' D(1 - F)^2}{2} + C_l D(1 - F) \left[ \frac{1 - \beta_0}{2} \right]. \quad \text{(24)}$$
8. With the exception of the use of \( C_h \) instead of \( C_h' \), (24) is identical to (14), so the equations for \( T^* \) and \( F^* \) are the same as in Equations (15) and (16) with the replacement of \( C_h' \) by \( C_h' \):

\[
T^* = \sqrt{\frac{2C_o}{DC_h} \left[ \frac{C_h + C_h'}{C_h'} \right] - \frac{[C_i(1 - \beta_0)/2]^2}{C_hC_h'}}
\]  
(25)

\[
F^* = \frac{C_i[(1 - \beta_0)/2] + C_h'T^*}{T^*(C_h + C_h')}
\]  
(26)

9. The necessary condition for the optimality of partial backordering (Equation (18)) is, as for the equations for \( T^* \) and \( F^* \), the same except for the replacement of \( C_h' \) by \( C_h' \):

\[
\beta_0 \geq \beta_0^* = 1 - 2\sqrt{2C_oC_hD/(DC_i)}.
\]  
(27)

10. Finally, the simplified equation for the optimal cost (Equation 19)) is the same except for replacing \( C_h' \) by \( C_h' \):

\[
\Gamma^* = \Gamma(T^*, F^*) = C_hDT^*F^*.
\]  
(28)

The process for determining the optimal solution described in Section 2.4 is identical except for the relevant equation numbers and not having to compute the value of \( C_h' \).

4. **Examples**

To illustrate we will use examples based on the example in Pentico et al. (2009).

4.1 **Example for the EPQ-PBO**

The parameter values are:

\[
D = 1100 \quad P = 9200 \quad \beta_0 = 0.70
\]
\(C_o = 275 \quad C_h = 2.00 \quad C_b = 3.20 \quad C_l = 4.00\)

(Note that, as shown in the example in Pentico et al. (2009), with these parameter values, \(\beta^*\) for a constant \(\beta\) is 0.7654, so \(\beta = 0.70\) would not result in using the EPQ with partial backordering.)

From these values we compute:

\[C'_h = C_h(1 - D/P) = 2.00(1 - 1100/9200) = 1.76087\]

\[C'_b = C_b\left[\beta_0 + \frac{(1 - \beta_0) - (1 + \beta_0)^2 D}{4P}\right] = (3.20)\left[0.70 + \frac{(1 - 0.70) - (1 + 0.70)^2 (1100)}{4(9200)}\right]\]

\[= 2.28357\]

\[\beta^*_0 = 1 - \frac{2\sqrt{2C_o C'_h D}}{DC'_l} = 1 - \frac{2\sqrt{2(275)(1.76087)(1100)}}{(1100)(4.00)} = 1 - 0.4692 = 0.5308\]

Since \(\beta_0 = 0.70 > 0.5308 = \beta^*_0\), using the EPQ with partial backordering costs less than using the EPQ with no backordering, so we compute the values of \(T\), \(F\), and \(I(T,F)\).

\[T = \sqrt{\frac{2C_o \left[C'_h + C'_b\right]}{DC'_h} - \frac{[C_l(1 - \beta_0)/2]^2}{C'_h C'_b}}\]

\[= \sqrt{\frac{2(275)}{(1100)(1.76087)}\left[1.76087 + 2.28357\right] - \frac{[4.00(1 - 0.70)/2]^2}{(1.76087)(2.28357)}} = 0.64294\]

\[F = \frac{C_l[(1 - \beta_0)/2] + C'_b T}{T[C'_h + C'_b]} = \frac{4.00[(1 - 0.70)/2] + (2.28357)(0.64294)}{(0.64294)(1.76087 + 2.28357)} = 0.79536\]

The cost of using the EPQ with partial backordering with these values of \(T\) and \(F\) is:

\[I(T,F) = C'_h DTF = (1.76087)(1100)(0.64294)(0.79536) = 990.50\text{ per year.}\]
Since this is less than the cost of not stocking the item, \( C_iD = (4.00)(1100) = 4400 \), using the EPQ with partial backordering with those values for \( T \) and \( F \) is optimal.

### 4.2 Example for the EOQ-PBO

The parameter values are the same except that there is no value for \( P \).

\[
C'_b = C_b \left[ \beta_0 + \frac{(1 - \beta_0)}{3} \right] = (3.20) \left[ 0.70 + \frac{(1 - 0.70)}{3} \right] = 2.5600
\]

\[
\beta^*_0 = 1 - \frac{2\sqrt{2C_oC'_bD}}{DC'_h} = 1 - \frac{2\sqrt{2(275)(2.00)(1100)}}{(1100)(4.00)} = 1 - 0.5000 = 0.5000
\]

Since \( \beta_0 = 0.70 > 0.5000 = \beta^*_0 \), using the EOQ with partial backordering costs less than using the EOQ with no backordering, so we compute the values of \( T, F, \) and \( I(T,F) \).

\[
T = \sqrt{\frac{2C_o}{DC_h} \left[ \frac{C_h + C'_b}{C'_b} \right]^2 - \frac{[C_i(1 - \beta_0)/2]^2}{C_hC'_b}} = \sqrt{\frac{2(275)}{(1100)(2.00)} \left[ \frac{2.00 + 2.56}{2.56} \right] - \frac{[(4.00)(1 - 0.70)/2]^2}{(2.00)(2.56)}} = 0.61237
\]

\[
F = \frac{C_i[(1 - \beta_0)/2] + C'_bT}{T[C_h + C'_b]} = \frac{4.00[(1 - 0.70)/2] + (2.56)(0.61237)}{(0.61237)[2.00 + 2.56]} = 0.77627
\]

\[
I(T,F) = C'_hDTF = (2.00)(1100)(0.61237)(0.77627) = 1045.81
\]

Since this is less than the cost of not stocking the item, \( C_iD = (4.00)(1100) = 4400 \), using the EOQ with partial backordering with those values for \( T \) and \( F \) is optimal.

### 5. Suggestions for Future Work

An important aspect of the development of any model is the recognition of its limitations.

With this in mind, we identify two general areas for possible extensions of our work in this
paper, both of which can be called sensitivity analysis. In its classic sense, sensitivity analysis means determining the effect(s) of changing the value(s) of one or more of a model’s parameters. In a broader sense, sensitivity analysis means determining the effect of changing the model structure itself. In this section we indicate where these ideas have arisen in the production management literature, and how they could profitably inform future research.

5.1 Classic sensitivity analysis

A number of researchers have addressed the issue of sensitivity analysis for the basic EOQ with partial backordering, although for the most part such analysis has focused on model response to changes in $\beta$ alone, rather than other parameters. For example, Park (1982) showed, as part of his example problem, that an increase in $\beta$ led to increases in the optimum total stockout level and order quantity and a decrease in the average cost per period. Chu and Chung (2004) concluded that the results of Park’s approach “are questionable since different conclusions may be made if different sets of numerical examples are analyzed.” They used an analytic approach to show that Park’s conclusions about the order quantity and cost are correct, but his conclusion about the maximum stockout level is only true if $C_v/C_b$ is less than a certain value that is based on $\beta^*$. Their analysis, however, was limited to cases in which $\beta^* \geq 0$. Yang (2007) extended Chu and Chung’s analysis to include the case in which $\beta^* < 0$. Leung (2009), using the model in Montgomery et al. (1973) instead of that in Park (1982), conducted an analysis similar to those by Chu and Chung (2004) and Yang (2007), determining the range for $\beta$ within which the decision variables and cost are monotonically increasing or decreasing.

More general approaches were undertaken by Borgonova and Peccati (2007) and Borgonovo (2008), who, rather than focusing on the effects of perturbations in only one
parameter, proposed and illustrated approaches to determining the relative importance of simultaneous changes in or uncertainty about the values of all of a model’s parameters. Borgonova and Peccati considered the basic EOQ model and a modification of the basic EOQ model that considers financing policies and the effects of a temporary sale, and Borgonova considered the same modified basic EOQ model.

While the analytic nature of our work lends itself to a variety of approaches for sensitivity analysis, the associated computational development is beyond the scope of this paper. Nevertheless, leveraging the above literature to assess the sensitivity of our results would be a valuable step in understanding the value of our work as a practical, real-world tool. In particular, it would help establish the utility of our results in the face of inaccurate parameter estimates.

5.2 Robustness under model changes

Defining an admissible class of models is the first step in understanding the robustness of a set of conclusions under changes in underlying model structure. Sometimes small changes in the model structure can lead to quite significant changes in the results. An example relevant to the basic EOQ with partial backordering can be found in Park (1982), who stated that “a numerical example shows that making the assumption of all backorders (or lost sales) when in fact a mixture of the two exists can significantly affect inventory costs.” A possible extension of our work here is to compare the results of using a linearly changing $\beta(t)$ model with those from using the exponential model or the rational model, neither of which has a closed-form solution. The objective would be to try to establish conditions (such as the rate at which $\beta(t)$ changes with respect to $t$) under which the linear model is a good approximation to the other two.
6. Conclusion

We have extended Pentico et al.’s (2009) model for the EPQ with partial backordering and a constant $\beta$ to allow the percentage of demand backordered to increase linearly as the time until delivery decreases. We found that, with appropriate redefinition of the unit holding, backorder, and lost sales costs, the objective function for the extended model has the same basic form as the objective function for the original model and, therefore, the equations for the optimal values of the decision variables also have the same basic forms as in the original model. As for the original model, we developed an equation for the minimum initial value of $\beta$ for which partial backordering costs less than using the classic EPQ model with no backordering.

We also showed how this model can be easily modified to determine the optimal policy for the EOQ with partial backordering if the percentage of demand backordered increases linearly as the time until delivery decreases.

This contribution is important not just because the increasing-$\beta$ models may more accurately reflect the realities of the market, but because it shows how our approach to establishing EOQ or EPQ optimality can be adapted to more general situations with minimal analytic modification. Identifying broad classes of functions for time-sensitive $\beta$ for which our analytic approach can be used to establish the existence of a global optimizer is the subject of continuing investigation.

References


Table 1

Notation for EOQ and EPQ with Backordering Rate Linearly Dependent on the Time to Delivery

Parameters:

\[ D = \text{demand per year} \]
\[ P = \text{production rate per year} \]
\[ s = \text{the unit selling price} \]
\[ C_o = \text{the fixed cost of placing and receiving an order} \]
\[ C_p = \text{the variable cost of producing a unit} \]
\[ C_h = \text{the cost to hold a unit in inventory for a year} \]
\[ C_b = \text{the cost to keep a unit backordered for a year} \]
\[ C_g = \text{the goodwill loss on a unit of unfilled demand} \]
\[ C_l = (s - C_p) + C_g = \text{the cost for a lost sale, including the lost profit on that unit and any goodwill loss} \]
\[ \beta(t) = \text{the fraction of stockouts that will be backordered at time } t \]

Variables:

\[ B(t) = \text{the backorder level at time } t \]
\[ B = \text{the maximum backorder position, with } \bar{B} \text{ being the average backorder level over the year} \]
\[ I(t) = \text{the inventory level at time } t \]
\[ I = \text{the maximum inventory level, with } \bar{I} \text{ being the average inventory level over the year} \]
\[ L = \text{total lost demand per year} \]
\[ S = \text{the maximum stockout level, including both backorders and lost sales} \]
\[ F = \text{the fill rate or the percentage of demand that will be filled from stock} \]
\[ T = \text{the time between orders or the length of an order cycle} \]
Diagram Showing Inventory Level and Time Intervals for the EPQ-PBO with Linearly Increasing Percentage of Backordering

Figure 1
Diagram Showing Inventory Level and Time Intervals for the EOQ-PBO with Linearly Increasing Percentage of Backordering

Figure 2